Compensated Demand Function and Income and Substitution

We can revisit the utility function $U(x,y) = xy^2, \forall\, x, y\ \geq 0$. We can derive the indirect utility function as a function of p_x, p_y, and M.

$$Max\ xy^2$$

$$s.t. p_x x + p_y y = M$$

Our Lagrangian is $\mathcal{L} = xy^2 - \lambda\left(p_x x + p_y y - M\right)$

Kuhn-Tucker Conditions:

1) $y^2 - p_x\lambda = 0 \Rightarrow \lambda = \frac{y^2}{p_x}$
2) $2xy - p_y\lambda = 0 \Rightarrow \lambda = \frac{2xy}{p_y}$
3) $p_x x + p_y y - M \leq 0$
4) $x, y \geq 0$

Combining (1) and (2) yields $y = \frac{2p_x x}{p_y}$. Substituting this into (3) gives us $x^*(p_x, p_y, M) = \frac{m}{3p_x}$. Substituting this back into (3) gives us $y^*(p_x, p_y, M) = \frac{2m}{3p_y}$.

Substituting $x^*$and y^* gives us $U^* = U|_{x=x^*, y=y^*} = \frac{4m^3}{27p_x p_y^2}$.

We can now derive the generalized compensated demand functions as functions of p_x, p_y, and U.

$$\mathrm{Min}\ \mathrm{p_x x + p_y y + \lambda(U - xy^2)}$$

$$\mathrm{s.t. U = xy^2}$$

Our Lagrangian is $\mathcal{L} = \mathrm{p_x x + p_y y + \lambda(U - xy^2)}$

Kuhn-Tucker Conditions:

1) $p_x - \lambda y^2 = 0 \Rightarrow \lambda = \frac{p_x}{y^2}$
2) $p_y - \lambda 2xy = 0 \Rightarrow \lambda = \frac{p_y}{2xy}$
3) $\mathrm{U - xy^2 \leq 0}$
4) $\mathrm{x, y \geq 0}$

Combing (1) and (2) yields $\mathrm{y = 2x\frac{p_x}{p_y}}$.

Substituting this into (3) yields $x_c^*(p_x, p_y, U) = (U)^{\frac{1}{3}}\left(\frac{p_y}{2p_x}\right)^{\frac{2}{3}}$.

Substituting this back into (3) yields $y_c^*(p_x, p_y, U) = \left(2U\frac{p_x}{p_y}\right)^{\frac{1}{3}}$.

Using the generalized compensated demand functions we just derived we can derive the expenditure function as functions as p_x, p_y, and U.

$$e^* = p_x x_c^* + p_y y_c^* = \left[2^{-2/3} + 2^{1/3}\right]\left(p_x p_y^2 U\right)^{1/3}$$

Suppose $p_x = \$2$, $p_y = \$3$, and $= \$200$.

The utility maximization consumption bundle at these prices and income are $x^*(2,3,200) = \frac{m}{3p_x} = \frac{200}{6} = \frac{100}{3}$ and $y^*(2,3,200) = \frac{2m}{3p_y} = \frac{400}{9}$.

If we now let p_x increase to \$4, leaving $p_y = \$3$, and $= \$200$ we can find the income and substitution effects of the price change.

Recall the $x^*(2,3,200) = \frac{200}{3 \cdot 2} = \frac{100}{3}$

$$U^* = U|_{x=x^*, y=y^*} = \overline{U}(2,3,200) = \frac{4 \cdot 200^3}{27 \cdot 2 \cdot 3^2} = 65843.6214$$

$$x_c^*(4,3,200) = (200)^{\frac{1}{3}}\left(\frac{3}{2 \cdot 4}\right)^{\frac{2}{3}} = 20.9987$$

Substitution effect is shown by the movement along the Hicksian demand:

$$x^*(2,3,200) - x_c^*(4,3,200) = \frac{100}{3} - 20.9987 = 12.3346$$

Substitution effect leads to a decrease of 12.3346 units of good x

$$m^*(4,3,200) = \left[2^{-\frac{2}{3}} + 2^{\frac{1}{3}}\right](4 \cdot 3 \cdot 200)^{\frac{1}{3}} = 251.9842$$

$$x^*(4,3,200) = \frac{200}{3 \cdot 2} = \frac{200}{12} = 16.6667$$

Income effect is shown by the movement along the Engel curve:

$$x_c^*(2,3,200) - x^*(4,3,200) = 20.9987 - 16.6667 = 4.3320$$

Income effect leads to a decrease of 4.3320 units of good x

The income necessary to maintain the original level of utility at the higher prices is $m^*(4, 3, 200) = \left[2^{-\frac{2}{3}} + 2^{\frac{1}{3}}\right](4 \cdot 3 \cdot 200)^{\frac{1}{3}} = 251.9842.$

The subsidy needed to restore the consumer to that original indifference curve is $S^* = m^*(4, 3, 65843.6214) - 200 = 251.9842 - 200 = 51.9842.$

The compensated loss of consumer's surplus associated with the price increase is $\Delta CS_c = S^* = \int_2^4 x_c^*\left(p_x, \overline{p_y}, \overline{U}\right) dp_x = \int_2^4 (\overline{U})^{\frac{1}{3}} \left(\frac{\overline{p_y}}{2p_x}\right)^{\frac{2}{3}} dp_x = 3\overline{U}^{1/3} \left(\frac{\overline{p_y}}{2}\right)^{2/3} p_x^{1/3}\Big|_2^4 =$ 51.9842

The uncompensated loss of consumer's surplus associated with the price increase is $\Delta CS_u = \int_2^4 x^*\left(p_x, \overline{p_y}, \overline{m}\right) dp_x = \int_2^4 \frac{\overline{m}}{3p_x} dp_x = \frac{m}{3}\left[\ln p_x + c\right]_2^4 = 46.2098$

The $\Delta CS_c > CS_u$ which implies this is a normal good.