Cost Functions

Assuming r is the rental rate for capital and w is the wage for labor we can find the long run expansion path for the production function $x = \frac{10KL}{K+L}$.

$$Min\ wL + rK$$

$$s.t.\ x = \frac{10KL}{K+L}$$

Our Lagrangian is $\mathcal{L} = wL + rK - \lambda\left(x - \frac{10KL}{K+L}\right)$

Kuhn-Tucker Conditions:

1) $w - \lambda\frac{10K^2}{(K+L)^2} = 0 \Rightarrow \lambda = w\frac{(K+L)^2}{10K^2}$
2) $r - \lambda\frac{10L^2}{(K+L)^2} = 0 \Rightarrow \lambda = r\frac{(K+L)^2}{10L^2}$
3) $x - \frac{10KL}{K+L} \leq 0$
4) $K, L \geq 0$

Combining (1) and (2) yields $K = \left(\frac{w}{r}\right)^{1/2} L$.

The long run production path for the production function $x = \frac{10KL}{K+L}$ as a function of r and w is $K = \left(\frac{w}{r}\right)^{1/2} L$.

The elasticity of substitution is derived by $\frac{d\left(\frac{K}{L}\right)}{d\left(\frac{w}{r}\right)}\frac{\frac{w}{r}}{\frac{K}{L}}$.

$$\sigma_{KL} = \frac{d\left(\frac{w}{r}\right)^{\frac{1}{2}}}{d\left(\frac{w}{r}\right)}\frac{\frac{w}{r}}{\left(\frac{w}{r}\right)^{1/2}} = \frac{\frac{1}{2}\left(\frac{w}{r}\right)^{-1/2}d\left(\frac{w}{r}\right)}{d\left(\frac{w}{r}\right)}\left(\frac{w}{r}\right)^{1/2} = \frac{1}{2}\left(\frac{w}{r}\right)^{-1/2}\left(\frac{w}{r}\right)^{1/2} = \frac{1}{2}$$

Suppose $w = \$1$ and $r = \$4$ and we would like to produce 10 units. We can easily find the cost minimizing input combination for the given production function.

Recall the long run expansion path is $K = \left(\frac{w}{r}\right)^{1/2} L$.

So by substituting in the wage and rental rates we obtain $K = \frac{1}{2}L$.

Substituting this and the desired output level into (3) we can derive $L^* = 3$.

Substituting this back into the long run expansion path we obtain $K^* = \frac{3}{2}$

The cost minimizing input combination for the given production function is $(3, \frac{3}{2})$.

Now we can derive the conditional input demands $K^*(r, w, x)$ and $L^*(r, w, x)$.

To do this we will substitute the derived long run expansion path into the production function and solve for the respective variable.

$$L^* = \frac{x(w^{1/2}+r^{1/2})}{10w^{1/2}}$$

$$K^* = \frac{x(w^{1/2}+r^{1/2})}{10r^{1/2}}$$

To derive the long-run total cost as a function of r, w, and x we substitute the conditional input demands $K^*(r, w, x)$ and $L^*(r, w, x)$ into our cost function.

$$LRTC = \frac{x(w^{1/2}+r^{1/2})^2}{10}$$

Recall the marginal cost function is derived by $\frac{\partial LRTC}{\partial x}$ and the average cost curve is derived by $\frac{\partial LRTC}{x}$.

With this particular production function $MC = ATC = \frac{(w^{1/2}+r^{1/2})^2}{10}$.

As in consumer theory, the λ from our Lagrangian in the minimization problem above has a particular interpretation.

By substituting our conditional input demands $K^*(r, w, x)$ and $L^*(r, w, x)$ into either (1) or (2) we obtain $\lambda = \frac{(w^{1/2}+r^{1/2})^2}{10}$.

This is the same as the marginal cost.

If we now assumer $r = \$4$ and $w = \$1$ we can find long run total, average, and marginal costs as function of output.

$$LRTC = \frac{9}{10}x$$

$$LRMC = \frac{9}{10}$$

$$LRAC = \frac{9}{10}$$

Suppose we continue to assume $r = \$4$ and $w = \$1$ but now fix $K = 4$.

We can now derive the short run variable cost, average cost, total cost, average total cost, and marginal cost as function of output.

By plugging into our restrictions into the proper equation from above we obtain the following:

$$x = \frac{40L}{4+L}$$
$$L = \frac{4x}{40-4x}$$
$$SRVC = \frac{4x}{40-4x}$$
$$SRAVC = \frac{4}{40-4x}$$
$$SRTC = \frac{4x}{40-4x} + 16$$
$$SRATC = \frac{4}{40-4x} + \frac{16}{x}$$

$$SRMC = \frac{160}{(40-4x)^2}$$

We can also prove that short run variable cost, average cost, total cost, and marginal cost take on equal values at the minimum short run average total cost.

$$Min\ \frac{4}{40-4x} + \frac{16}{x}$$

FOC: $\frac{16}{(40-4x)^2} - \frac{16}{\mathrm{x}^2} = 0$

$$\mathrm{x}^2 = (40-4x)^2$$
$$x = 8, 13\frac{1}{3}$$

$$x^* = 13\frac{1}{3}$$

$$SRATC = \frac{9}{10}$$
$$SRMC = \frac{9}{10}$$
$$LRATC = \frac{9}{10}$$
$$LRMC = \frac{9}{10}$$