What We've Got Here Is Failure to Conjugate

Dr. LaLonde

UT Tyler Math Club

November 29, 2017

When is a sequence of numbers random?

Consider the following two sequences of numbers:

$$0.0802, 0.7935, 0.2920, 0.6989, 0.5616, 0.3179, 0.4768,$$
$$0.9410, 0.2647, 0.3910, 0.1524, 0.1557, 0.1296, 0.3831,$$
$$0.0216, 0.0226, 0.4050, 0.1193, 0.0811, 0.7926, \ldots$$

and

$$0.3299, 0.6598, 0.6804, 0.6392, 0.7216, 0.5568, 0.8864,$$
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Which one is random? .

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Which one is random? The **first one**.

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In other words, it **avoids** the pattern 321.

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In other words, it **avoids** the pattern 321. A truly random sequence should eventually contain **all** patterns!

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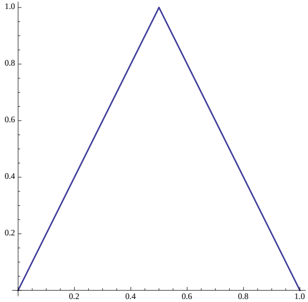

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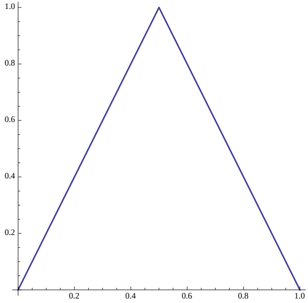

Patterns of dynamical systems

Suppose we have a map $f : [0,1] \to [0,1]$. The process of iterating f, i.e., choosing a point $x \in [0,1]$ and computing

$$x,\ f(x),\ f^2(x) = f(f(x)),\ f^3(x),\ \ldots$$

is an example of a (discrete) **dynamical system**.

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Definition

Let $f : [0,1] \to [0,1]$ and $x \in [0,1]$. The **pattern** *of f at x with length n, denoted by $\mathrm{Pat}(x, f, n)$, is the permutation of the set $\{1, 2, \ldots, n\}$ that is in the same relative order as the numbers*

$$x,\ f(x),\ f^2(x),\ f^3(x),\ \ldots,\ f^{n-1}(x).$$

Let's look back at the sequence generated by the tent map:

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Then the relative order of the first four terms is:

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Conjugacy

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Suppose $f, g : [0,1] \to [0,1]$*. We say that the associated dynamical systems are* **conjugate** *if there is a homeomorphism* $h : [0,1] \to [0,1]$ *such that*

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Since $h : [0,1] \to [0,1]$, it's enough for h to be strictly increasing or strictly decreasing and onto.

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Conjugacy: an example

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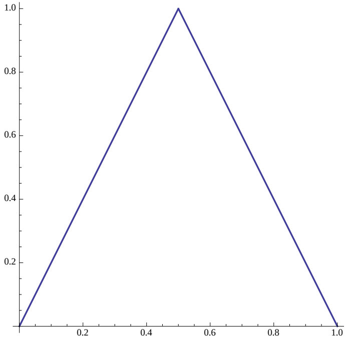

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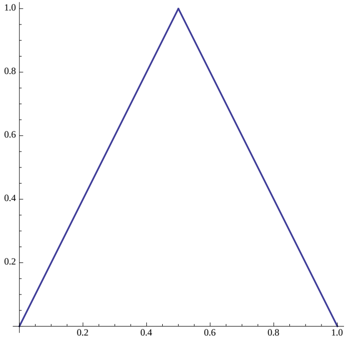

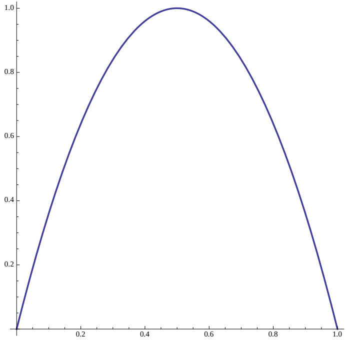

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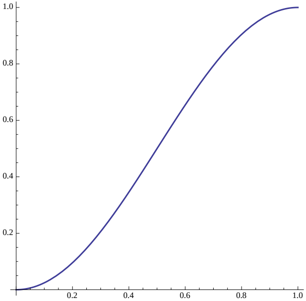

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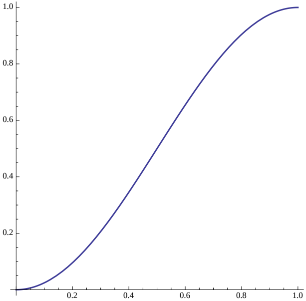

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$$\begin{aligned} L(h(x)) &= 4 \cdot \tfrac{1}{2}(1 - \cos \pi x) \cdot \tfrac{1}{2}(1 + \cos \pi x) \\ &= 1 - \cos^2 \pi x \\ &= \sin^2 \pi x \end{aligned}$$

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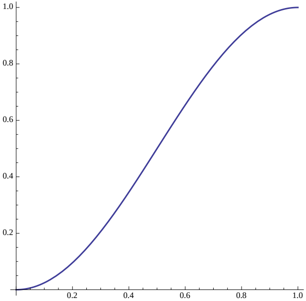

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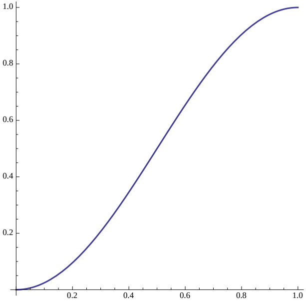

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when $0 \leq x \leq \frac{1}{2}$ and

$$\begin{aligned} h(T(x)) &= \tfrac{1}{2}(1 - \cos 2\pi(1 - x)) \\ &= \tfrac{1}{2}(1 - \cos 2\pi x) \\ &= \sin^2 \pi x \end{aligned}$$

when $\frac{1}{2} < x \leq 1$.

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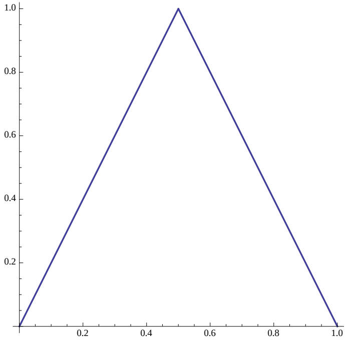

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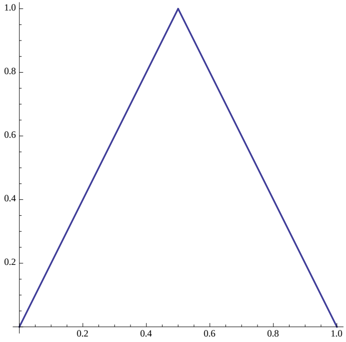

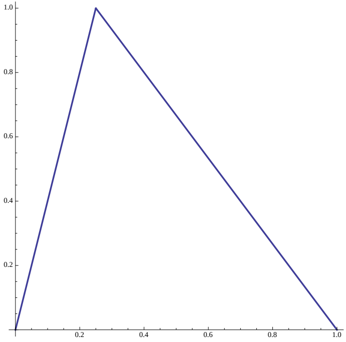

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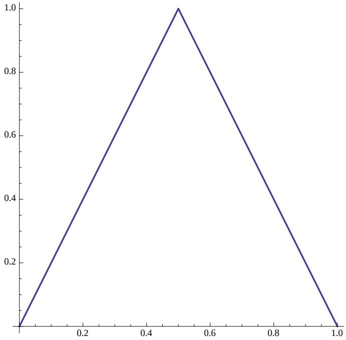

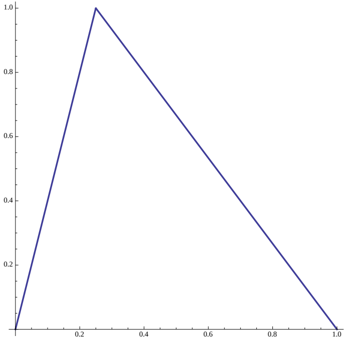

These maps are actually conjugate.

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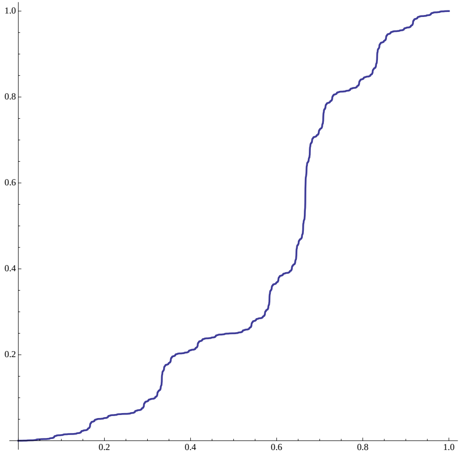

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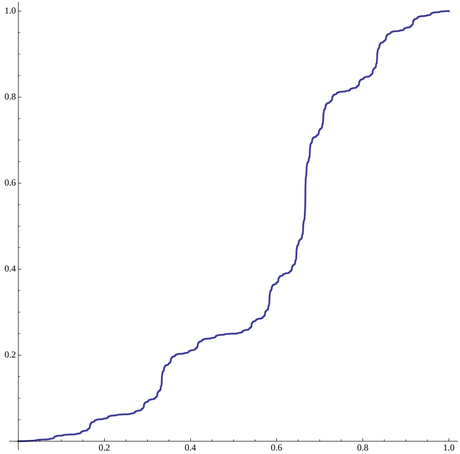

What function is this?

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Thus h satisfies the functional equation

$$\boxed{h(x) = \begin{cases} \frac{1}{4}h(2x) & \text{if } 0 \le x \le \frac{1}{2} \\ 1 - \frac{3}{4}h(2(1-x)) & \text{if } \frac{1}{2} < x \le 1 \end{cases}}$$

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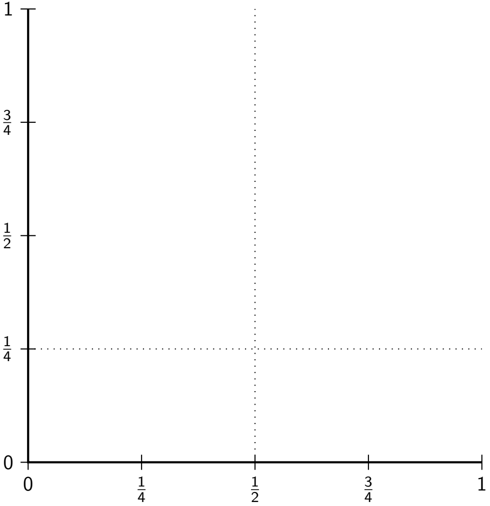

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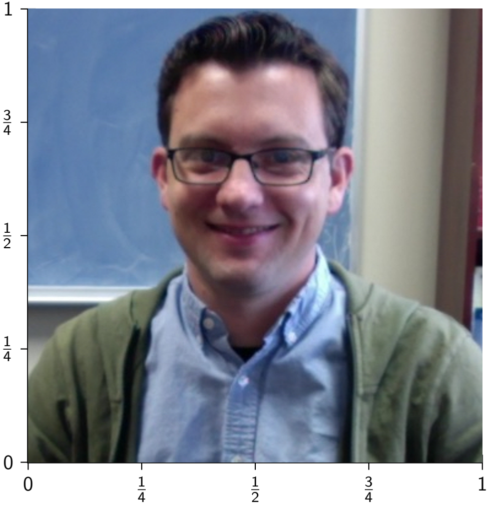

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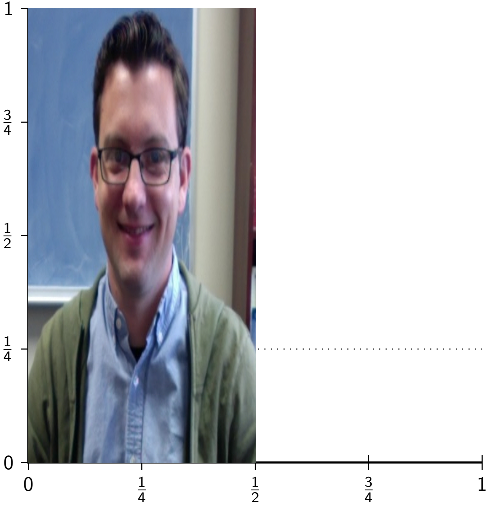

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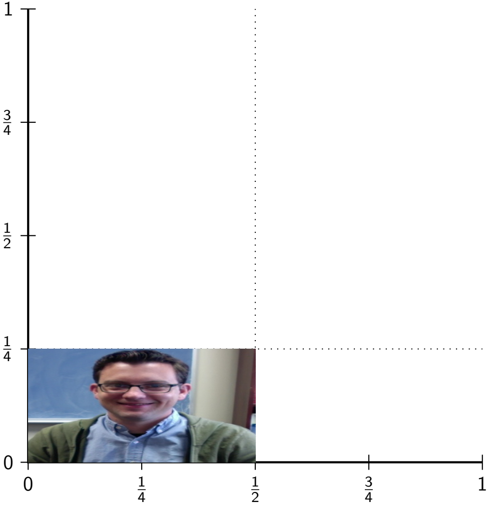

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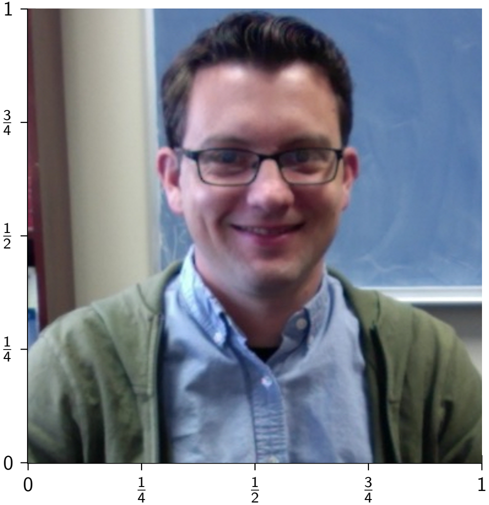

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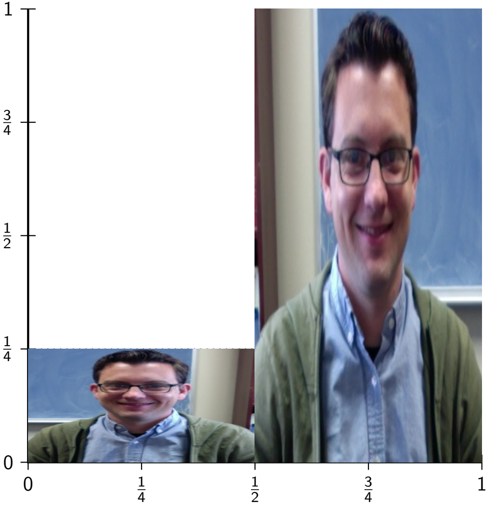

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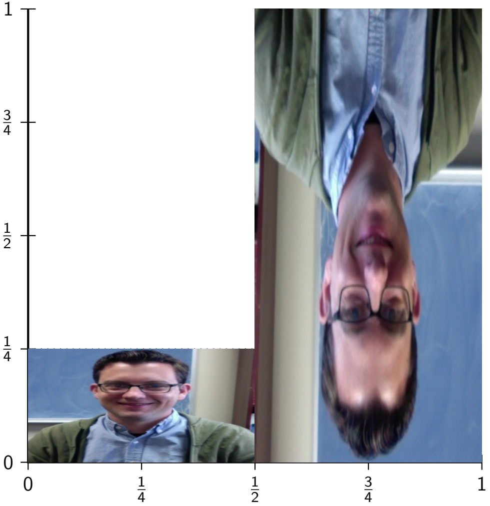

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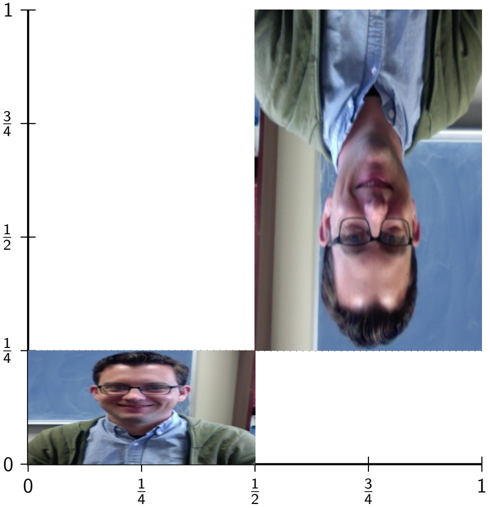

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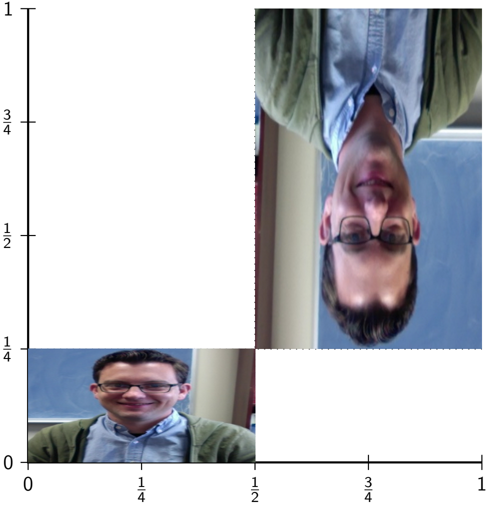

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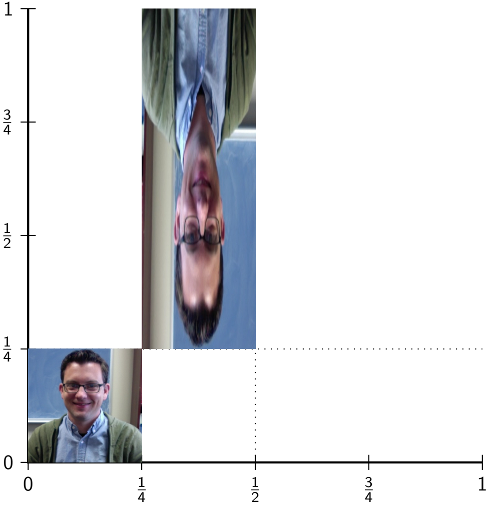

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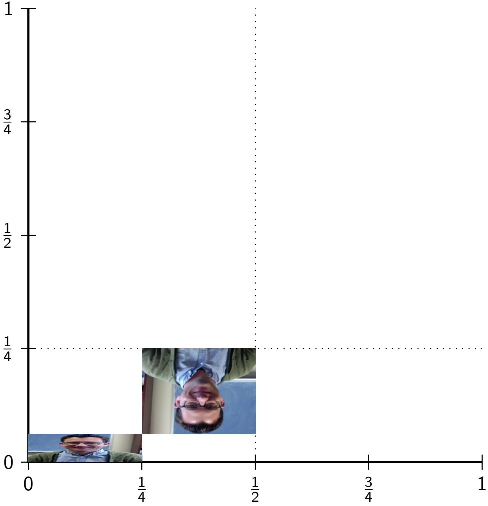

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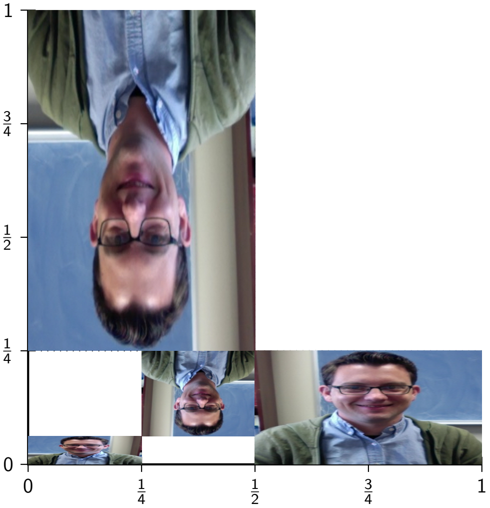

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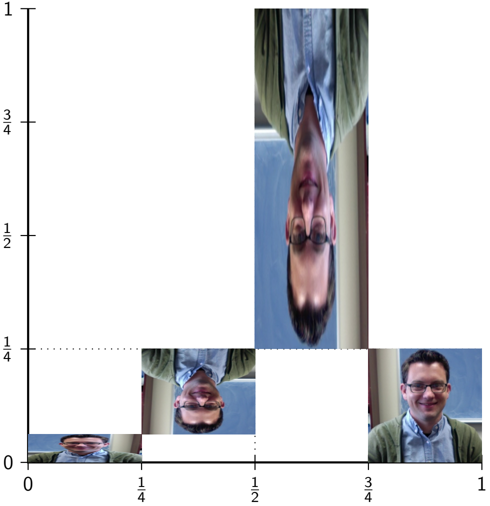

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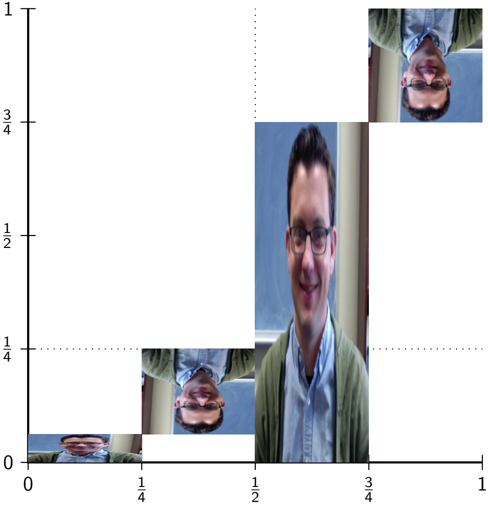

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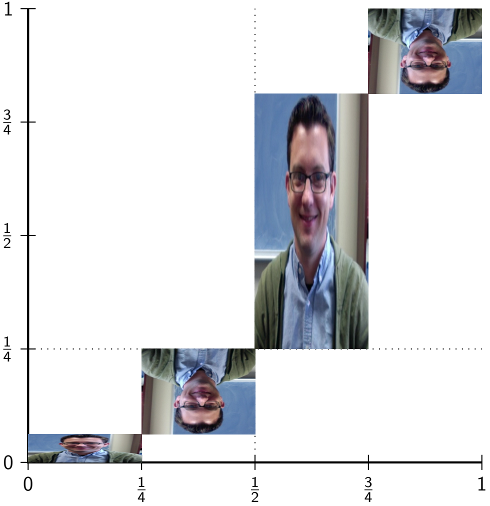

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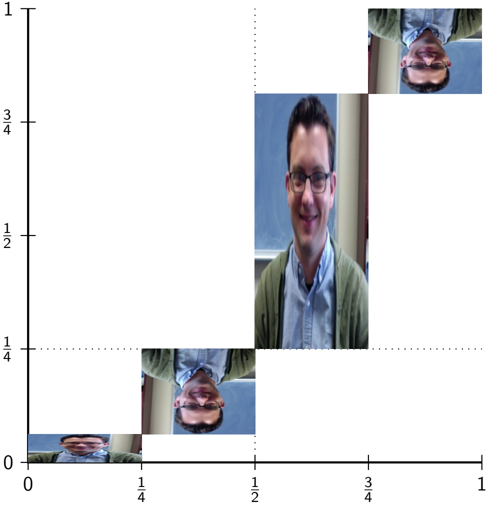

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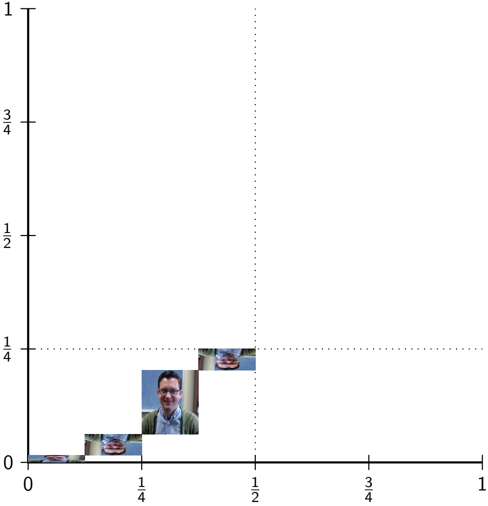

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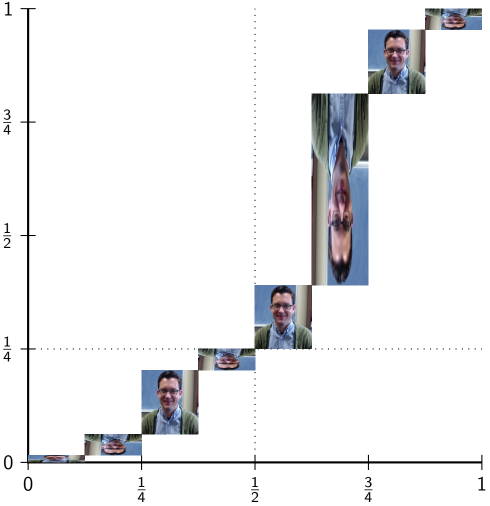

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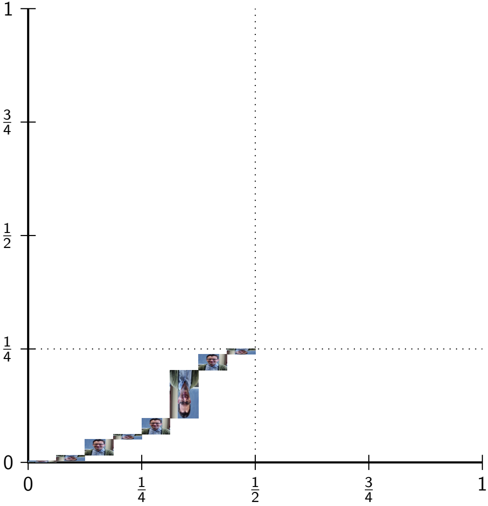

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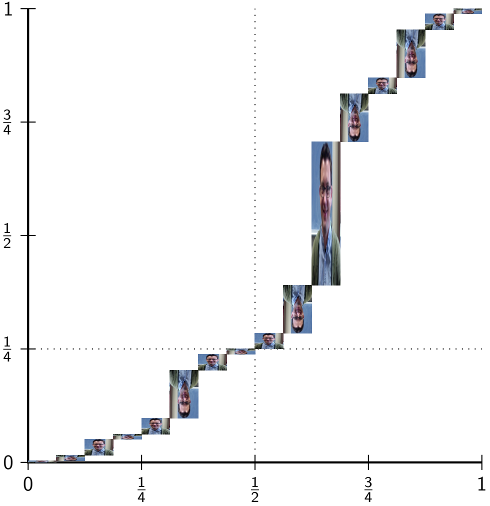

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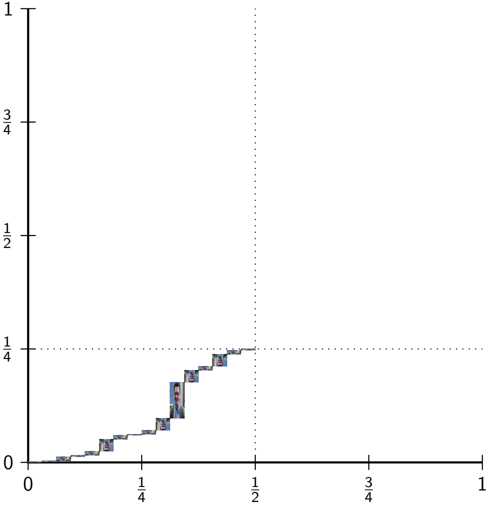

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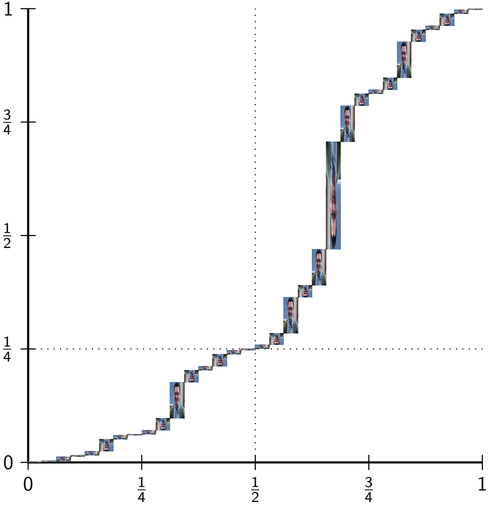

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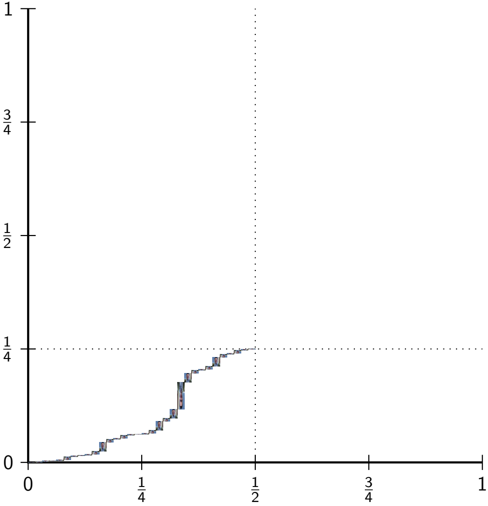

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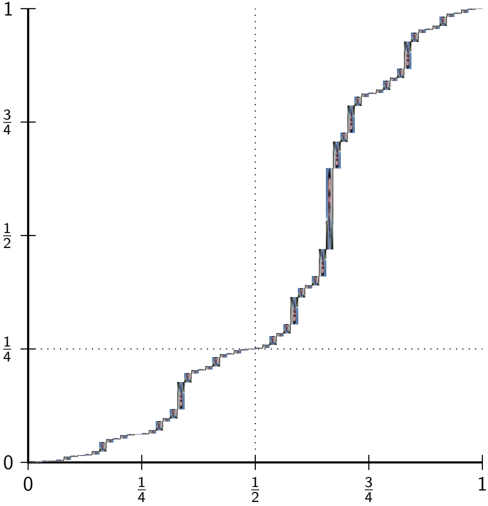

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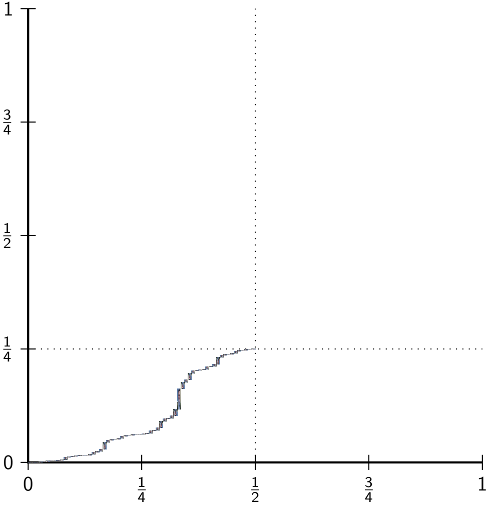

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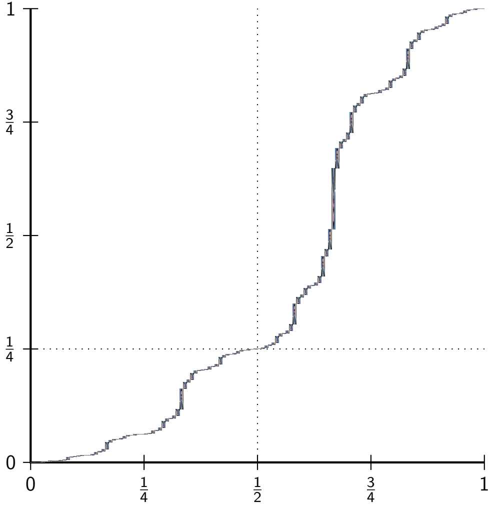

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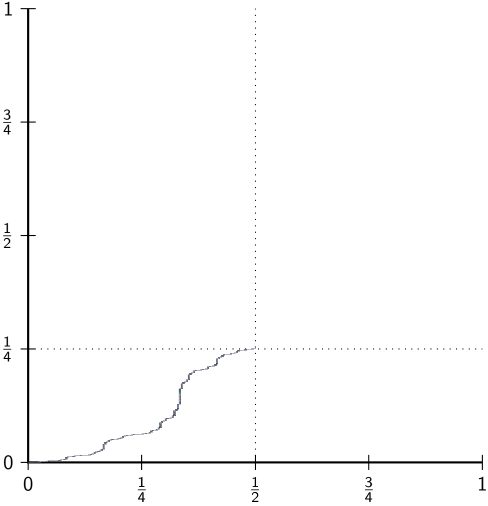

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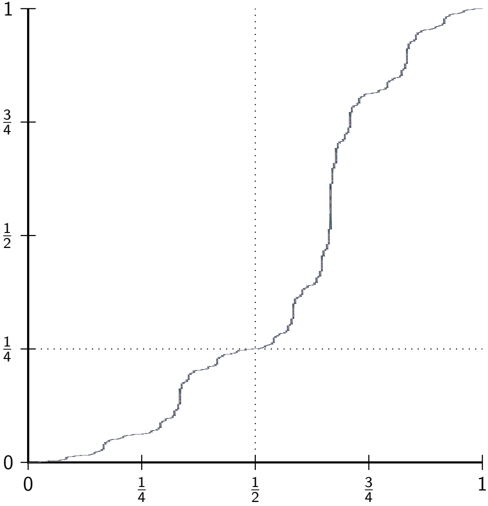

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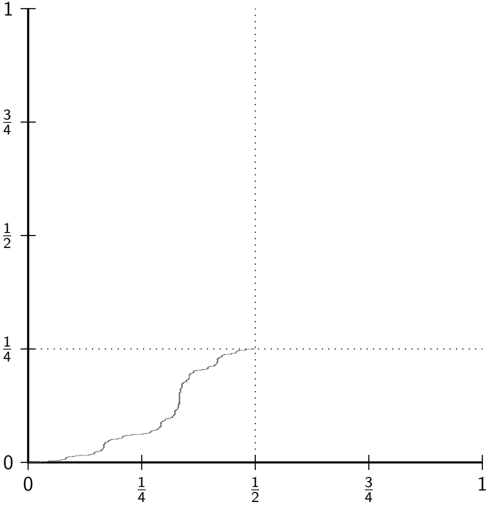

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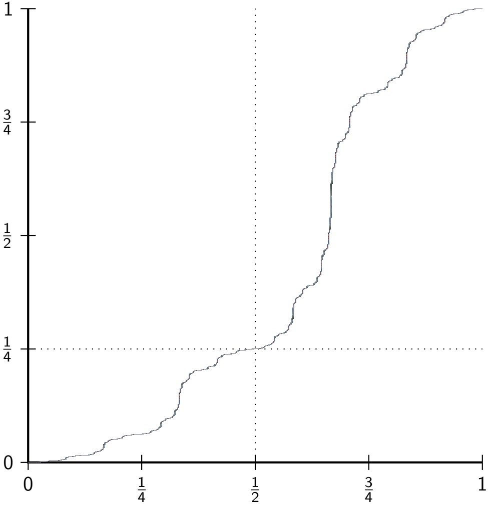
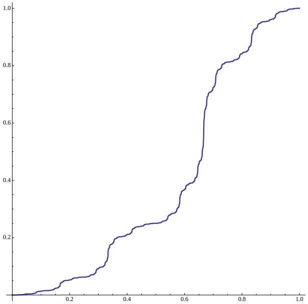
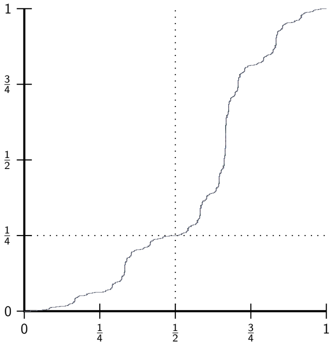


Compare to the graph from earlier:





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- This sequence converges uniformly to a function

$$h(x) = \lim_{n\to\infty} h_n(x)$$

by the Banach Fixed Point Theorem.

How does this help us study allowed patterns?

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Theorem

If f and g are conjugate, then $\mathrm{Allow}(f) = \mathrm{Allow}(g)$*.*

Non-conjugate dynamical systems

What if f and g have "similar" dynamics, but they're not conjugate?

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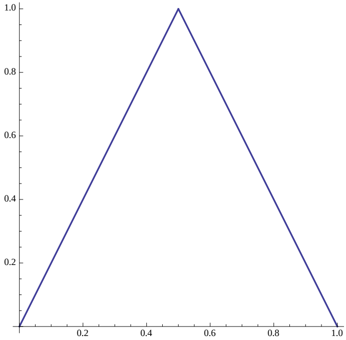
$$T(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2(1-x) & \text{if } \frac{1}{2} < x \leq 1 \end{cases}$$



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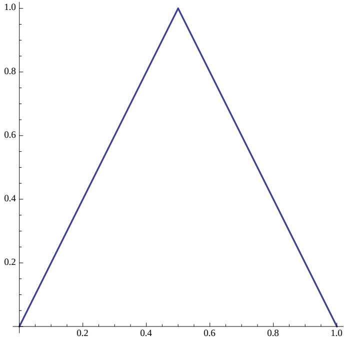

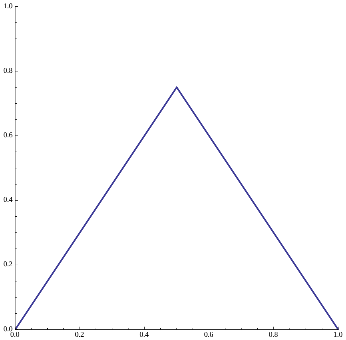

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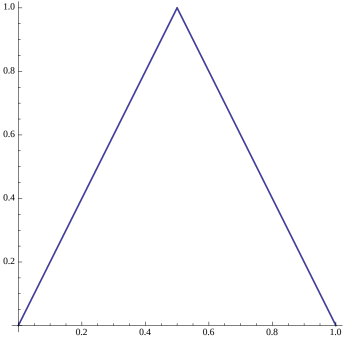

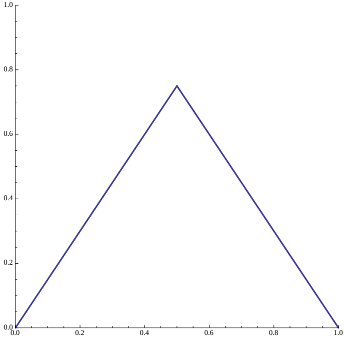

These maps are **not** conjugate.

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As in our earlier example, h must satisfy the functional equation

$$\boxed{h(x) = \begin{cases} \frac{1}{2}h\left(\frac{3}{2}x\right) & \text{if } 0 \leq x \leq \frac{1}{2} \\ 1 - \frac{1}{2}h\left(\frac{3}{2}(1-x)\right) & \text{if } \frac{1}{2} < x \leq 1 \end{cases}}$$

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Start with $h_0(x) = x$:

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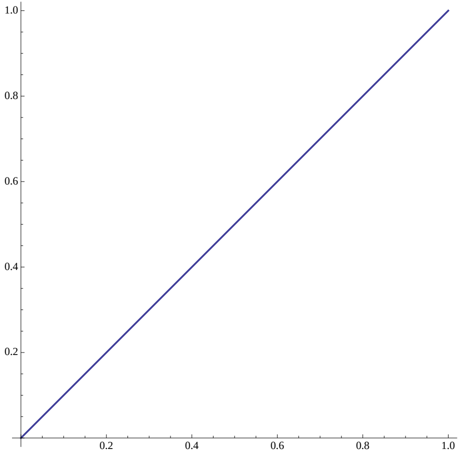

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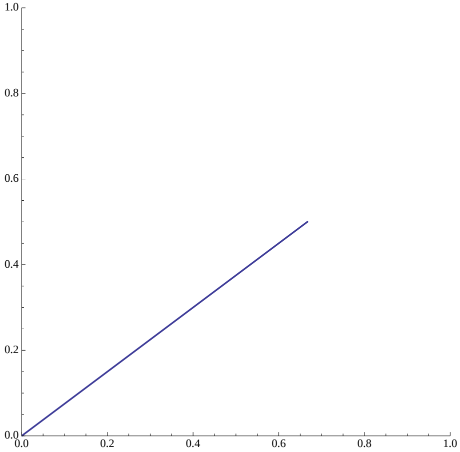

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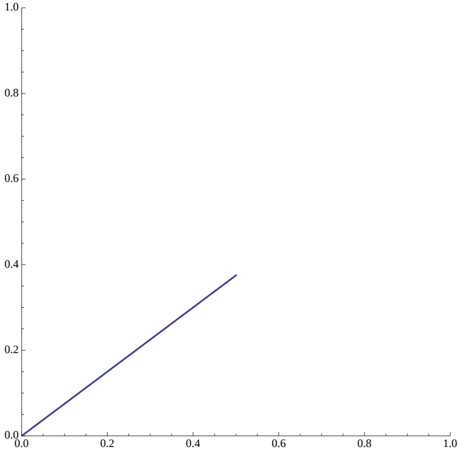

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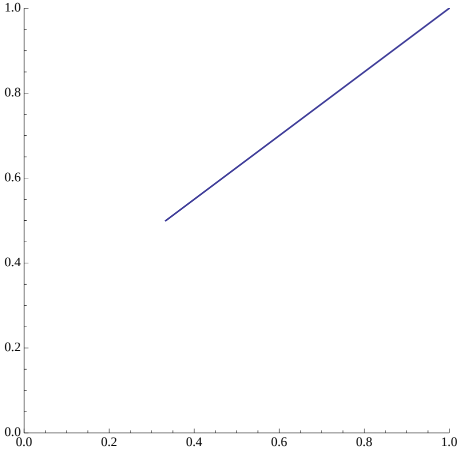

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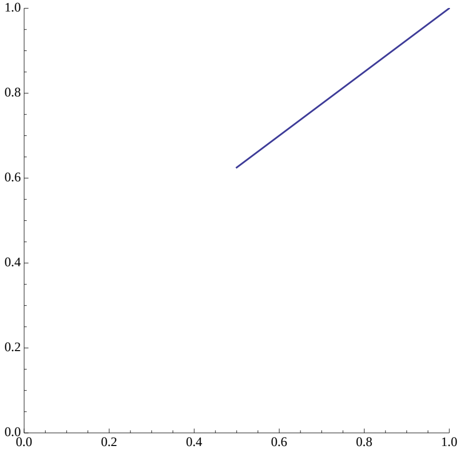

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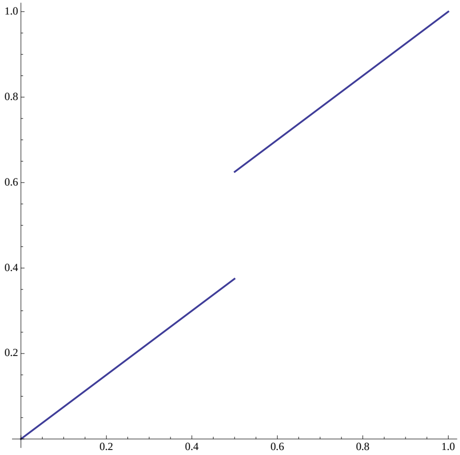

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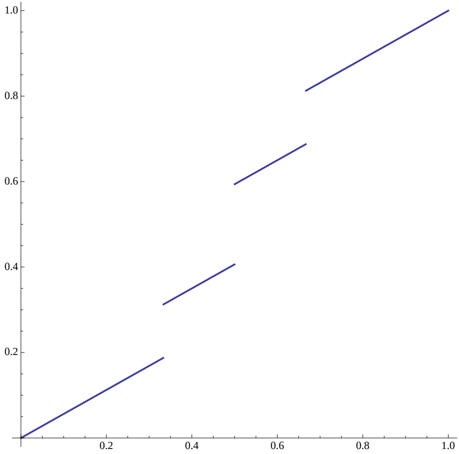

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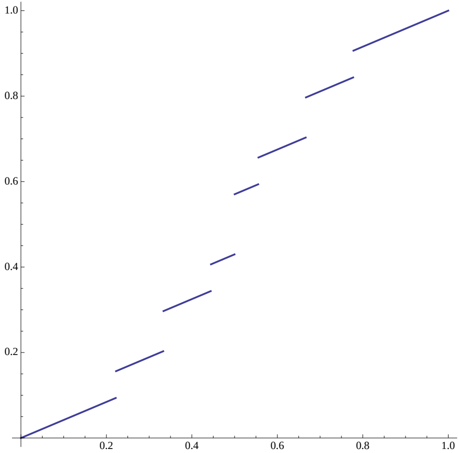

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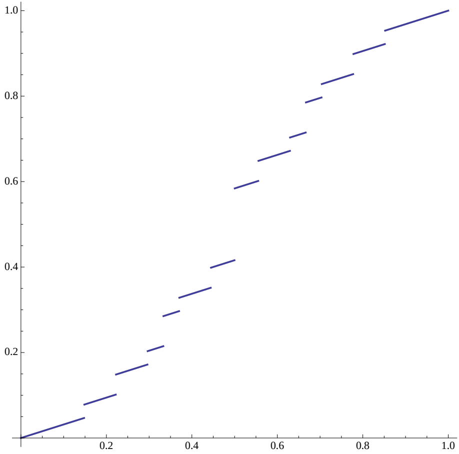

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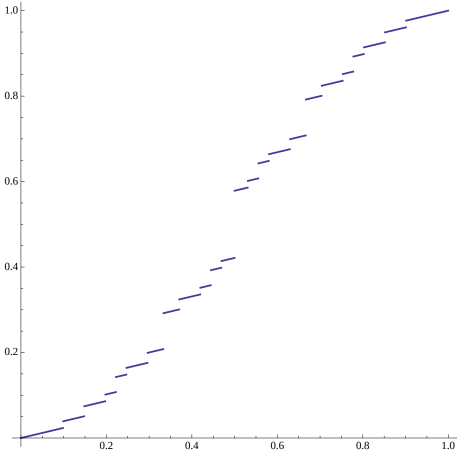

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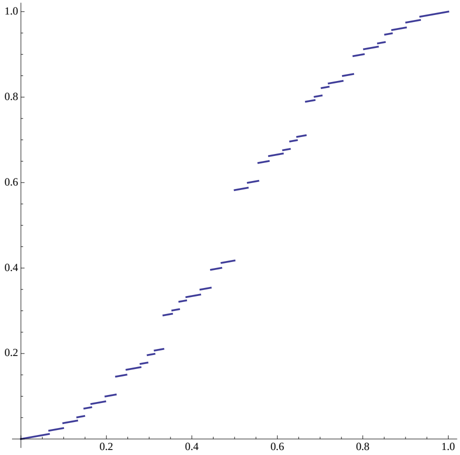

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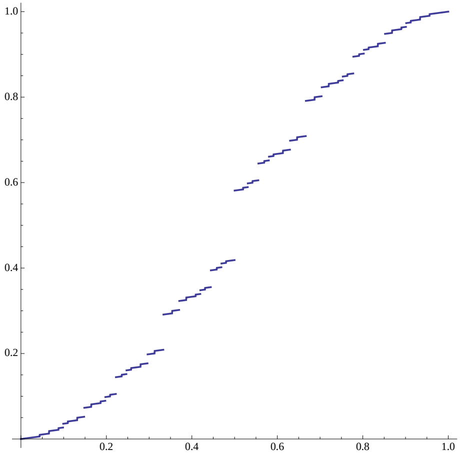

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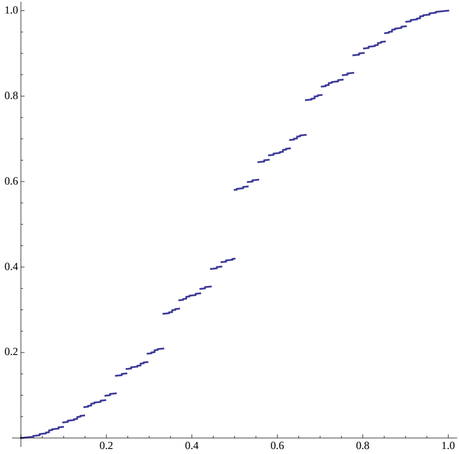

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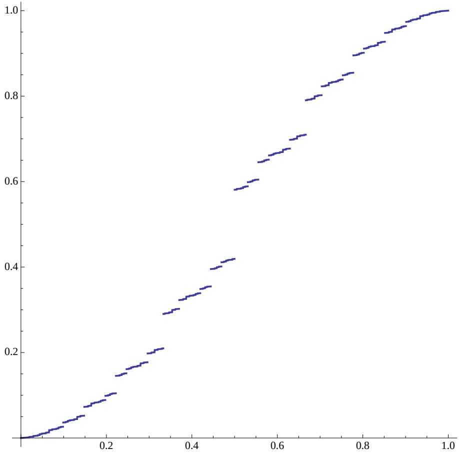

Commuters and allowed patterns

Notice that h isn't continuous or onto, but it is strictly increasing.

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But we can't go the other way, since h isn't onto.

The End

Some reading materials for those who are interested:

K. T. Alligood, T. D. Sauer, and J. A. Yorke, *Chaos: An Introduction to Dynamical Systems*, Springer, New York, 1996.

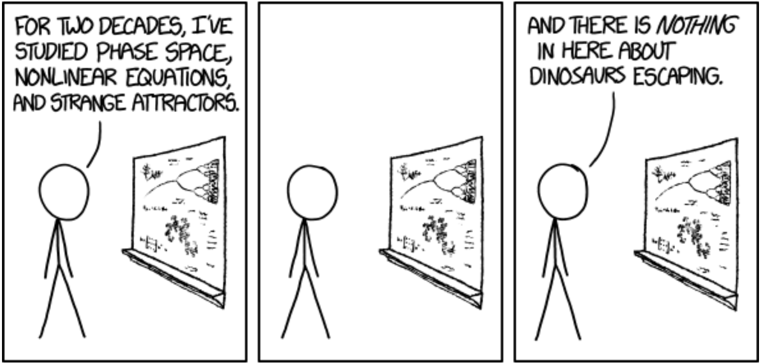
K. Archer and S. M. LaLonde, *Allowed patterns of symmetric tent maps via commuter functions*, SIAM J. Discrete Math, 31 (2017), pp. 317–334.

S. Elizalde and Y. Liu, *On basic forbidden patterns of functions*, Discrete Appl. Math, 159 (2011), pp. 1207–1216.

S. Elizalde and K. Moore, *Characterizations and enumerations of patterns of signed shifts*, arXiv:1711.05213v1, (2017).

J. D. Skufca and E. M. Bollt, *A concept of homeomorphic defect for defining mostly conjugate dynamical systems*, Chaos, 18 (2008).

The End



(Courtesy of *xkcd*)