Groupoids in Analysis

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Let G be a group. We will generally assume two things:

- G is equipped with a (Hausdorff) topology making multiplication and inversion continuous.
- This topology is locally compact.

We call such a G a **locally compact topological group**.

Example

Some examples:

- $\mathbb{R}$ (or $\mathbb{R}^n$) with the usual Euclidean topology.
- $\mathbb{Z}$ with the discrete topology.
- $\mathrm{GL}_n(\mathbb{R})$ with the topology inherited from $\mathbb{R}^{n^2}$.

Transformation Groups

Recall that a group G **acts** on a set X if there is a map

$$G \times X \to X, \quad (g, x) \mapsto g \cdot x$$

such that

1. $e \cdot x = x$ for all $x \in X$,
2. $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.

Assume that X is a locally compact Hausdorff space. If the map $(g, x) \mapsto g \cdot x$ is continuous, then we say that the action of G on X is continuous.

Definition

If G acts continuously on X, the pair (G, X) is called a **transformation group***.*

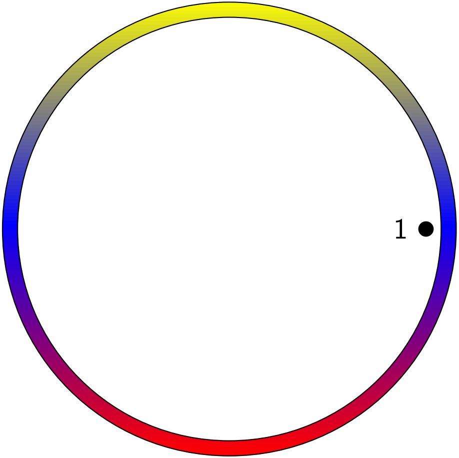
Example: Irrational rotation

- Let $X = \mathbb{T}$ be the unit circle in $\mathbb{C}$.
- Let $\theta \in [0, 1]$ be an irrational number.
- For $n \in \mathbb{Z}$ and $z \in \mathbb{T}$, define

$$n \cdot z = e^{2\pi i n\theta} z.$$

- This is a continuous action, so $(\mathbb{Z}, \mathbb{T})$ is a transformation group.
- What does the action actually look like?

$\theta = \frac{\pi}{90}, \; n = 0$



So...Where's the Analysis?

Let's assume that X is a compact Hausdorff space. People like me are interested in

$$C(X) = \{f : X \to \mathbb{C} : f \text{ is continuous}\}.$$

What sort of object is this?

- Vector space **and** ring $\implies$ $\mathbb{C}$-algebra.
- Norm: for $f \in C(X)$,

$$\|f\|_\infty = \sup_{x \in X} |f(x)|$$

$\implies$ (complete) metric space.

So...Where's the Analysis?

Now define $f^* \in C(X)$ by

$$f^*(x) = \overline{f(x)}.$$

Then $* : C(X) \to C(X)$ is an *involution* (i.e., $(f^*)^* = f$). Moreover:

$$\|f^* f\|_\infty = \sup_{x \in X} |f^*(x) f(x)| = \sup_{x \in X} |f(x)|^2 = \|f\|_\infty^2.$$

These properties together say that $C(X)$ is an example of a (commutative) C^*-**algebra**.

C^*-dynamical systems

Suppose that (G, X) is a transformation group with X compact.

- Then G acts on $C(X)$ as well: for $g \in G$, define α_g by

$$\alpha_g(f)(x) = f(g^{-1} \cdot x)$$

  for all $f \in C(X)$.
- Then α_g is an automorphism of $C(X)$.
- The map $\alpha : G \to \mathrm{Aut}(C(X))$ given by

$$g \mapsto \alpha_g$$

  is a continuous group homomorphism.

Definition

*The triple $(C(X), G, \alpha)$ is a C^*-**dynamical system**.*

Crossed products

- This is really a special case: if A is a C^*-algebra, G is a LC group, and $\alpha : G \to \mathrm{Aut}(A)$ is a continuous homomorphism, then we call (A, G, α) a C^*-dynamical system.
- Given a C^*-dynamical system, we can build a new C^*-algebra

  $$A \rtimes_\alpha G,$$

  called the **crossed product** of A by G. We won't discuss how.

But...what is it good for?

1. The crossed product encodes information about A, G, and the dynamics.
2. It gives an interesting way of building new examples of C^*-algebras.

Example: Irrational rotation algebras

Let $(\mathbb{Z}, \mathbb{T})$ be the transformation group associated to rotation by $2\pi\theta$, where θ is irrational.

- We get an action τ of $\mathbb{Z}$ on $C(\mathbb{T})$, and the crossed product

$$A_\theta = C(\mathbb{T}) \rtimes_\tau \mathbb{Z}$$

  is called an **irrational rotation algebra**.
- These were very popular in the 1980's, since they have some interesting properties:

  1. They are all **simple** (i.e., no nontrivial closed ideals).
  2. If $\theta_1, \theta_2 \in [0, 1/2]$, then $A_{\theta_1} \cong A_{\theta_2}$ iff $\theta_1 = \theta_2$.

Groupoids

A groupoid is essentially a group, but the multiplication is not defined everywhere. It consists of:

- A set G.
- A set $G^{(2)} \subset G \times G$ and a map $G^{(2)} \to G$ (multiplication) given by
$$(\gamma, \eta) \mapsto \gamma\eta.$$
- Multiplication is associative whenever the products "make sense."
- A map $G \to G$ given by $\gamma \mapsto \gamma^{-1}$ (inversion), where
$$\gamma^{-1}(\gamma\eta) = \eta \quad \text{and} \quad (\gamma\eta)\eta^{-1} = \gamma$$
whenever $(\gamma, \eta) \in G^{(2)}$.
- **Topological groupoid:** multiplication and inversion are continuous.

Groupoids (continued)

The role of the "identity" (or "unit") is played by the **unit space** of a groupoid:

$$G^{(0)} = \left\{ u \in G : u = u^{-1} = u^2 \right\}.$$

There are always two maps from G onto $G^{(0)}$, called the **range** and **source maps**:

$$r(\gamma) = \gamma\gamma^{-1} \quad \text{and} \quad s(\gamma) = \gamma^{-1}\gamma$$

We can think of $r(\gamma)$ and $s(\gamma)$ as left and right "identities" for γ:

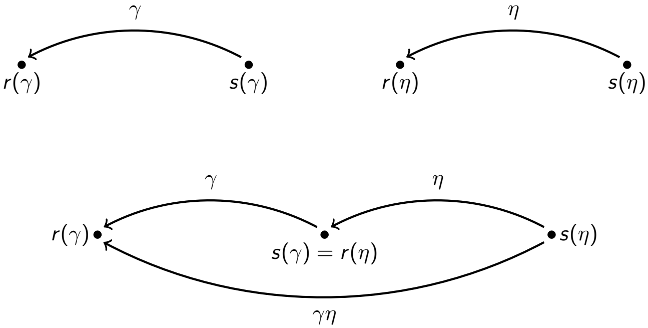
$$r(\gamma)\gamma = \gamma \quad \text{and} \gamma s(\gamma) = \gamma$$

Groupoids (continued)

The range and source maps tell us when two groupoid elements can be composed:

$$(\gamma, \eta) \in G^{(2)} \iff s(\gamma) = r(\eta)$$

We can even think of groupoid elements as "arrows" between units:



Groupoid Examples

Groupoids generalize lots of structures with which you are familiar.

- **Groups!** If G is a group, then

$$G^{(2)} = G \times G \quad \text{and} \quad G^{(0)} = \{e\}.$$

  Multiplication and inversion are the usual group operations.

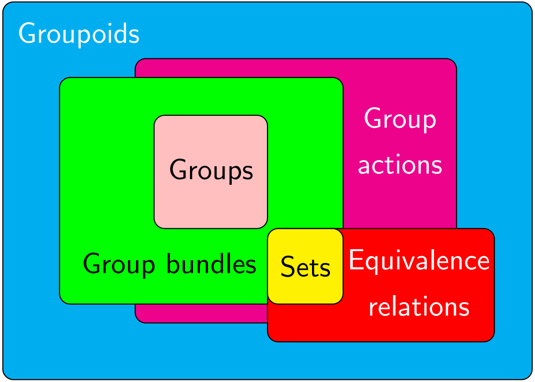
- **Sets!** If X is any set, then $X \times X$ is a groupoid:

$$\begin{aligned} \big((x,y),(z,w)\big) \in (X \times X)^{(2)} &\iff y = z \\ (x,y)(y,z) &= (x,z) \\ (x,y)^{-1} &= (y,x) \end{aligned}$$

  $X \times X$ is called the **pair groupoid** of X.

- **Equivalence relations!**

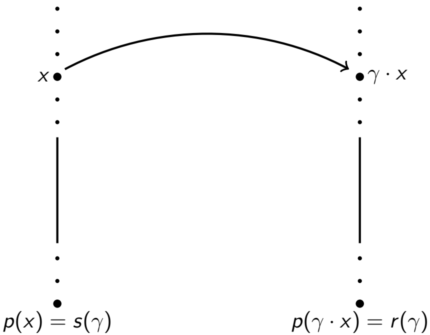
Groupoid Examples



Groupoid actions

As with groups, groupoids can act on sets (or topological spaces).

- The action is only "partially defined."
- Let X be a set with a surjective map $p : X \to G^{(0)}$. Then $\gamma \in G$ can only act on $x \in X$ if $p(x) = s(\gamma)$.



Groupoid crossed products

Groupoids can act on certain kinds of C^*-algebras, called $C_0(G^{(0)})$-algebras.

- If a locally compact Hausdorff groupoid acts on a C^*-algebra A via some action α, we call the triple (A, G, α) a **groupoid dynamical system**.
- As with groups, we can form a new C^*-algebra, called the **groupoid crossed product**:

$$A \rtimes_\alpha G.$$

- I'm interested in the following type of question: if A and G have "nice" properties, then does $A \rtimes G$ have similar "nice" properties?

Results

There are two properties that C^*-algebras can have which are very desirable: **nuclearity** and **exactness**. (Both are related to tensor products of C^*-algebras.)

Theorem (L., 2012)

If A is a nuclear C^*-algebra and G is an amenable groupoid, then $A \rtimes G$ is nuclear.

Theorem (L., 2013)

If A is an exact C^*-algebra and G is an amenable groupoid, then $A \times G$ is exact.

Both of these results generalize theorems for *group* crossed products that have been known for some time.

Thank you!

Questions?