Properties of Groupoid Dynamical Systems and their Crossed Products

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Overview

Groupoids generalize many mathematical objects, with groups being the foremost example. Just as groups classify global symmetries of objects, groupoids can be used to characterize *local* symmetries. When a groupoid G acts on a C^*-algebra A, one can construct a new C^*-algebra $A \rtimes G$, called the groupoid crossed product. This directly generalizes the crossed product of a C^*-algebra by a *group*, and it is natural to ask whether certain results carry over from groups to groupoids. We will discuss two such results here.

Nuclearity and Exactness

Nuclear C^*-algebras

Nuclearity is a very desirable property for a C^*-algebra to possess. Loosely speaking, it means that a C^*-algebra behaves well with respect to tensor products. If A and B are C^*-algebras, the *algebraic* tensor product

$$A \odot B$$

may carry more than one C^*-norm. Therefore, there are generally multiple ways to complete it to obtain a tensor product C^*-algebra. Most people care about the two extremes:

$$A \otimes_{\max} B \quad \text{and} \quad A \otimes_{\sigma} B,$$

the maximal and minimal tensor products. If the various tensor products coincide, then the situation is very nice.

Definition A C^*-algebra A is called **nuclear** if

$$A \otimes_{\max} B = A \otimes_{\sigma} B$$

for every C^*-algebra B.

Most reasonably nice C^*-algebras (such as commutative ones) are nuclear.

Exactness

A concept which is related to nuclearity is *exactness*. It also involves tensor products of C^*-algebras.

Definition A C^*-algebra A is **exact** if whenever

$$0 \to B \to C \to D \to 0$$

is a short exact sequence of C^*-algebras, the sequence

$$0 \to B \otimes_{\sigma} A \to C \otimes_{\sigma} A \to D \otimes_{\sigma} A \to 0$$

is also exact.

Since the sequence

$$0 \to B \otimes_{\max} A \to C \otimes_{\max} A \to D \otimes_{\max} A \to 0$$

is always exact, it follows that any nuclear C^*-algebra is automatically exact. However, there are exact C^*-algebras which are not nuclear.

Groupoids

Basics

Informally, a **groupoid** is a set G with a "partially defined" multiplication operation

$$(\gamma, \eta) \mapsto \gamma\eta$$

and an inversion map $G \to G$:

$$\gamma \mapsto \gamma^{-1}.$$

The **unit space** plays the role of the "identity":

$$G^{(0)} = \{u \in G : u = u^{-1} = u^2\}.$$

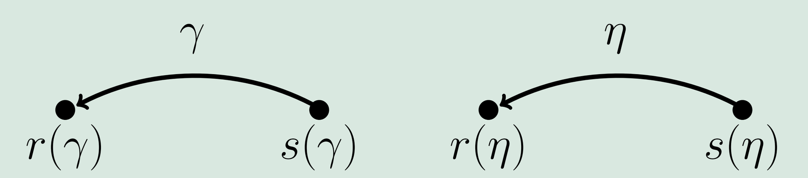
There are **range** and **source** maps $r, s : G \to G^{(0)}$:

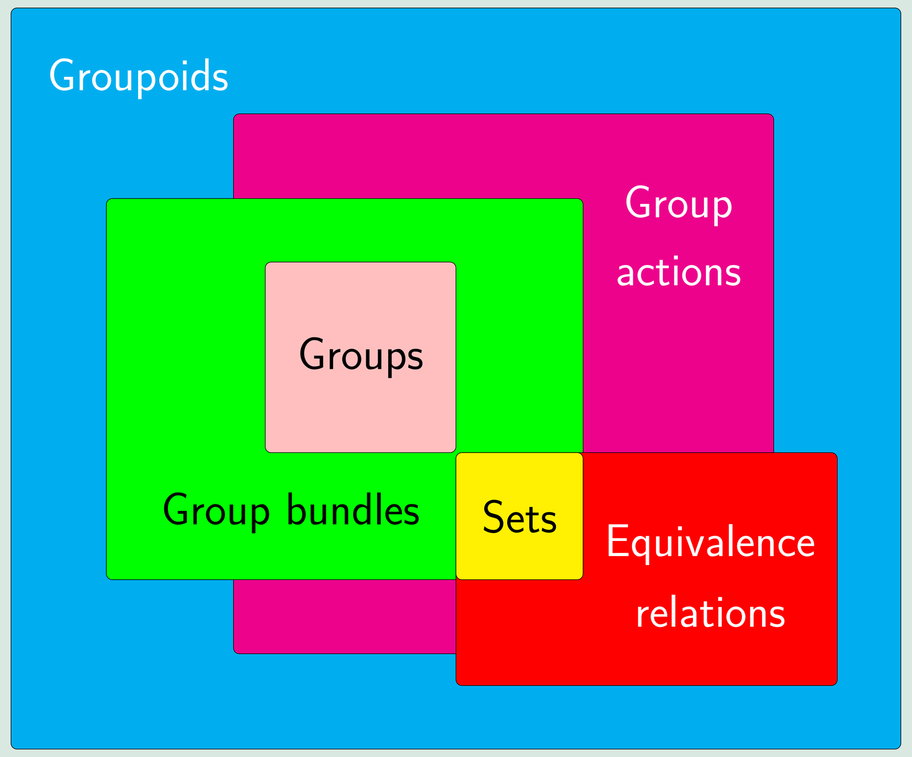
$$r(\gamma) = \gamma\gamma^{-1}, \quad s(\gamma) = \gamma^{-1}\gamma,$$

and

$$r(\gamma)\gamma = \gamma \quad \text{and} \quad \gamma s(\gamma) = \gamma \quad \forall \gamma \in G.$$

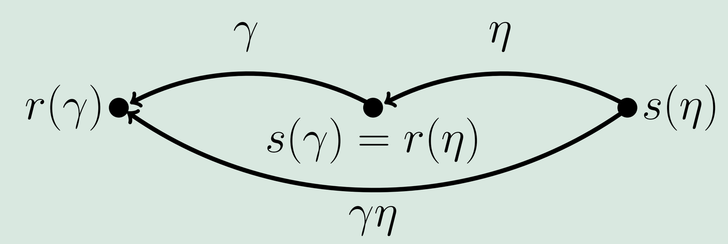
One can think of a groupoid as a group in which the product is only partially defined. Another convenient interpretation of a groupoid is a collection of "arrows" between units:





Groupoids generalize many different mathematical objects.

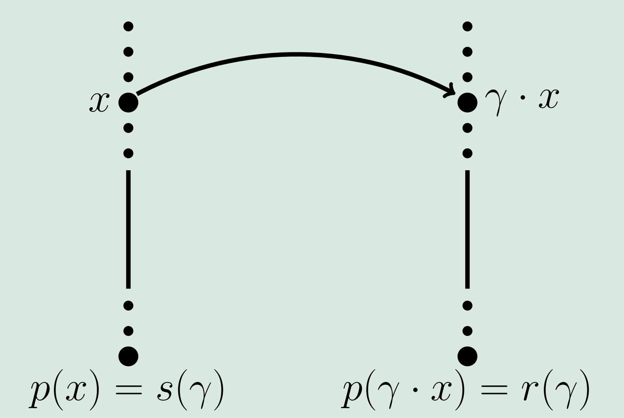
Note that γ, η are composable exactly when $s(\gamma) = r(\eta)$:



Groupoid Dynamical Systems and Crossed Products

Groupoid Actions

A groupoid G can act on a set X, provided that there is a surjection $p : X \to G^{(0)}$. The action is much like a group action, except it is only partially defined: if $\gamma \in G$ and $x \in X$, then $\gamma \cdot x$ is defined exactly when $s(\gamma) = p(x)$.



Dynamical Systems

Let G be a locally compact Hausdorff groupoid. For G to act on a C^*-algebra A, we require that A be a $C_0(G^{(0)})$-algebra. That is, there is a bundle $\mathcal{A}$ of C^*-algebras over $G^{(0)}$, and A can be viewed as a section algebra of $\mathcal{A}$. An **action** of G on A is just a family of isomorphisms

$$\alpha = \{\alpha_x\}_{x \in G}, \quad \alpha_x : \mathcal{A}_{s(x)} \to \mathcal{A}_{r(x)}.$$

The triple (A, G, α) is referred to as a **groupoid dynamical system**.

Crossed Products

Given a groupoid dynamical system (A, G, α), one can build a new C^*-algebra that encodes information about the system. We start by forming the pullback bundle

$$r^*\mathcal{A} \to G$$

and we consider the space of continuous compactly supported sections

$$\Gamma_c(G, r^*\mathcal{A}).$$

We assume that G carries a **Haar system** – a family of measures which plays the role of the Haar measure on a locally compact group. Using these measures, it's possible to define a convolution-like product and an involution which make $\Gamma_c(G, r^*\mathcal{A})$ into a $*$-algebra. We can define a norm on it by considering an appropriate collection of $*$-representations of $\Gamma_c(G, r^*\mathcal{A})$ on Hilbert space. The completion is a C^*-algebra,

$$A \rtimes_{\alpha} G,$$

called the **crossed product** of A by G.

The groupoid crossed product directly generalizes the idea of a crossed product by a *group* (which in turn generalizes group C^*-algebras). Indeed, if G is a locally compact group, then the groupoid crossed product associated to G agrees with the usual one.

Main Results

It was shown by Philip Green in [2] that if A is a nuclear C^*-algebra and G is an amenable *group*, then $A \rtimes_{\alpha} G$ is nuclear. We present the analogue for groupoids here. It requires measurewise amenability for groupoids, which is a technical, measure-theoretic condition.

Theorem 1 (L., 2012) If (A, G, α) is a groupoid dynamical system with A nuclear and G amenable, then the crossed product $A \rtimes_{\alpha} G$ is nuclear.

It is also known [4] that if A is exact and G is an amenable group, then $A \rtimes_{\alpha} G$ is an exact C^*-algebra. The following analogue holds for groupoids.

Theorem 2 (L., 2013) If (A, G, α) is a groupoid dynamical system with A exact and G amenable, then the crossed product $A \rtimes_{\alpha} G$ is exact.

Applications

Groupoids and their C^*-algebras have many potential uses.

- Groupoids generalize many sorts of objects, so they provide a very general mathematical framework in which to work.
- Groupoids can be used to analyze symmetries of objects. In particular, one may be able to recover more information with groupoids instead of groups.
- It has recently been proposed that groupoid crossed products may be useful in time-frequency analysis. Guillemard and Iske [3] have suggested that crossed products of AF-algebras by certain groupoids, coupled with persistent homology, could be used to examine the structure of data coming from a given signal.

References

[1] Geoff Goehle, *Groupoid crossed products*, Ph.D. thesis, Dartmouth College, May 2009.

[2] Philip Green, *The local structure of twisted covariance algebras*, Acta Math. **140** (1978), 191–250.

[3] Mijail Guillemard and Armin Iske, *On groupoid C^*-algebras, persistent homology, and time-frequency analysis*, Preprint, September 2011.

[4] Eberhard Kirchberg, *Commutants of unitaries in UHF algebras and functorial properties of exactness*, J. Reine Angew. Math. **452** (1994), 39–77.

[5] Dana P. Williams, *Crossed products of C^*-algebras*, Mathematical Surveys and Monographs, no. 134, American Mathematical Society, Providence, RI, 2007.