Research Statement

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Broadly speaking, I am a functional analyst, and my research interests lie primarily in the study of operator algebras and C^*-algebras associated to various kinds of algebraic, topological, and combinatorial objects. I am specifically interested in the C^*-algebras used to model groups, groupoids, inverse semigroups, directed graphs, and dynamical systems. These types of C^*-algebras are built in such a way that their structures reflect those of the underlying objects, often through ideals, modules, and representation theory. Such properties of C^*-algebras are well-understood, so it is often fruitful to study an object through the lens provided by its associated C^*-algebra.

Most of my independent work has centered on C^*-algebras arising from *groupoid dynamical systems*, which involve an action of a groupoid G on a topological space, or more generally on a C^*-algebra A. Given such an action, one can build a new C^*-algebra $A \rtimes G$, called the *crossed product*, which encodes information about A, G, and the relevant dynamics. Beginning with my doctoral dissertation, I have written several papers that help determine when these crossed products have certain structural properties that are important to the classification of C^*-algebras. Along the way, I have investigated some intrinsic properties of groupoids (particularly *exactness*) that help to ensure the associated C^*-algebras are well-behaved. I have also worked on developing extensions of results for crossed products to more general structures known as *Fell bundles*.

In addition to my solo work, I have co-authored two papers with David Milan in which we brought some powerful results for groupoid C^*-algebras to bear on inverse semigroups. We obtained several structural results, including so-called *uniqueness theorems* for inverse semigroup C^*-algebras and a correspondence between the ideals of an inverse semigroup and those of its C^*-algebra. Many of our results and the conditions we imposed to obtain them were inspired by well-known concepts for directed graphs and their C^*-algebras. In the same vein, I also have a continuing project with a former undergraduate student in which we have studied connections between directed graphs and groupoids.

Aside from my work on operator algebras, I am also interested in classical dynamical systems and their applications to problems in enumerative combinatorics. I have published one paper on this topic with Kassie Archer, in which we studied the allowed patterns of a one-parameter family of tent maps. We are currently collaborating on some related problems for shift maps, and we are particularly interested in the likelihood of obtaining certain allowed patterns for these maps.

1 | Background on C^*-algebras and Groupoids

Suppose A is a $\mathbb{C}$-algebra equipped with an involution $* : A \to A$ and a submultiplicative norm. We say that A is a *C^*-algebra* if it is complete with respect to its norm, and the norm satisfies the *C^*-identity*, meaning that $\|a^*a\| = \|a\|^2$ for all $a \in A$. Two fundamental examples of C^*-algebras are the algebra $C(X)$ of continuous functions on a compact Hausdorff space X (with the supremum norm and the involution given by pointwise complex conjugation) and the algebra $B(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$ (with the operator norm and the involution given by the adjoint).

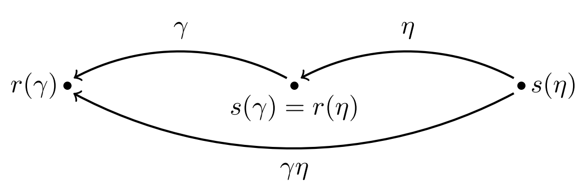
In addition to these prototypical examples, C^*-algebras also arise naturally in the field of harmonic analysis. Given a locally compact group G, one can build a C^*-algebra $C^*(G)$ from the continuous compactly-supported functions on G via convolution. (If G is a finite group, $C^*(G)$ is simply the familiar group algebra $\mathbb{C}[G]$.) The utility of this construction is that the representation theory of G is mirrored in that of $C^*(G)$, and representations of C^*-algebras are very well-understood. These ideas have been generalized to *C^*-dynamical systems*, where a group G acts continuously on a C^*-algebra A via automorphisms. From a dynamical system we can construct a new C^*-algebra $A \rtimes G$, called the *crossed product*, which encodes information about A, G, and the associated group action. A comprehensive account of crossed products can be found in [Wil07].

Much of my work involves generalizations of these ideas to the realm of *groupoids*. Loosely speaking, a groupoid is like a group, except multiplication is only partially defined. While this idea may seem odd at first, groupoids are quite useful because they can be used to simultaneously study many different mathematical objects, including groups, group actions, topological spaces, directed graphs, and equivalence relations.

To be more precise, a *groupoid* consists of a set G, a set $G^{(2)} \subseteq G \times G$ of "composable pairs", and a multiplication map $G^{(2)} \to G$. Multiplication is associative whenever all the relevant products are defined, and each $\gamma \in G$ has an inverse γ^{-1} (which allows for cancellation) and left and right identities $r(\gamma)$ and $s(\gamma)$, called the *range* and *source* respectively. The range and source always belong to a special subset of G, called the *unit space*:

$$G^{(0)} = \{u \in G : u = u^2 = u^{-1}\},$$

It often helps to think of a groupoid element as an arrow from from source to range, which actually gives a nice criterion for two groupoid elements to be composable: $(\gamma, \eta) \in G^{(2)}$ if and only if $s(\gamma) = r(\eta)$.



This picture fits nicely with the alternative, concise definition of a groupoid as a "small category with inverses". Under this interpretation, the objects are precisely the units, and each groupoid element represents an isomorphism from one object to another.

2 | Groupoid Crossed Products

Suppose G is a locally compact Hausdorff groupoid and A is a C^*-algebra. Given a continuous action α of G on A, we call the triple (A, G, α) a *groupoid dynamical system*. From a dynamical system, one can construct a new C^*-algebra $A \rtimes_\alpha G$, called the *crossed product of A by G*. As in the group case, the crossed product encodes information about A, G, and the associated dynamics. Groupoid crossed products are discussed extensively in [MW08] and [Goe09].

It is quite natural to ask if imposing certain conditions on A and G will yield desirable properties for the crossed product $A \rtimes G$. My initial investigation into groupoid crossed products focused on two particular structural properties of C^*-algebras, called *nuclearity* and *exactness*. Both properties are fairly technical, but they guarantee that a C^*-algebra is well-behaved in certain ways. Consequently, they often appear as crucial hypotheses in classification results for C^*-algebras.

Nuclearity and exactness are both defined in terms of tensor products of C^*-algebras, which are inherently complicated since there are multiple ways to reasonably define the tensor product of two C^*-algebras A and B. There are two extreme cases that are most often studied, namely the *minimal* and *maximal tensor products*, denoted by $A \otimes_\sigma B$ and $A \otimes_{\max} B$, respectively. These two tensor products each have advantages—the maximal one has good functorial properties, while the minimal one is easier to describe in a concrete way. Naturally, operator algebraists are interested in identifying situations where the two tensor products coincide. For certain C^*-algebras, which we term *nuclear*, this is always the case.

Definition. A C^*-algebra A is said to be *nuclear* if we have $A \otimes_\sigma B = A \otimes_{\max} B$ for every C^*-algebra B.

In [LaL15], I was able to give conditions that guarantee a groupoid crossed product is nuclear. The crucial hypotheses are that A is nuclear and G has a technical property called *measurewise amenability.*

Theorem 1 (LaLonde)**.** *If (A, G, α) is a separable groupoid dynamical system with A nuclear and G measurewise amenable, then $A \rtimes_\alpha G$ is nuclear.*

This result directly generalizes a theorem of Green [Gre78], though several technical hurdles that are not present in the group case must be overcome when working with groupoids.

We noted earlier that the maximal tensor product has good functorial properties. One such example is the fact that if A is a fixed C^*-algebra, then the functor $A \otimes_{\max} \_\_$ is exact. That is, given a short exact sequence $0 \to I \to B \to B/I \to 0$ of C^*-algebras,

$$0 \to A \otimes_{\max} I \to A \otimes_{\max} B \to A \otimes_{\max} B/I \to 0$$

is still exact. The situation is bleaker for the minimal tensor product—it need not preserve short exact sequences. Therefore, we single out the C^*-algebras for which $A \otimes_\sigma \_\_$ is exact.

Definition. A C^*-algebra A is *exact* if whenever $0 \to I \to B \to B/I \to 0$ is a short exact sequence of C^*-algebras, the sequence

$$0 \to A \otimes_\sigma I \to A \otimes_\sigma B \to A \otimes_\sigma B/I \to 0$$

is also exact.

In [Kir94, Proposition 7.1(v)], Kirchberg gave conditions that guarantee crossed products by groups are exact. I generalized this result to the groupoid setting in [LaL15].

> **Theorem 2** (LaLonde)**.** *Let (A, G, α) be a separable groupoid dynamical system with A exact and G measurewise amenable. Then $A \rtimes_\alpha G$ is exact.*

These theorems and some of the technical results used to prove them have found further use in papers on K-theory [BD19, OO20], amenability [Kra22], and exact groupoids [AD21].

2.1 | Fell Bundles

Operator algebraists have recently taken an interest in a far-reaching generalization of groupoid crossed products known as *Fell bundles*. The definition of a Fell bundle is quite technical, and the C^*-algebras of Fell bundles can be somewhat difficult to work with. However, if one can prove theorems about Fell bundle C^*-algebras, then the payoff is huge—any such result will immediately "trickle down" to all the other types of objects that are modeled by Fell bundles, including groupoid crossed products, *twisted* crossed products, and many other constructions. As a result, Fell bundles provide a unified framework in which one can study many different types of C^*-algebras at once.

Recently, it was shown in [IKSW18] that Fell bundles are not all that far removed from groupoid crossed products. In particular, any Fell bundle C^*-algebra is Morita equivalent to a certain groupoid crossed product. This so-called "Stabilization Theorem" gives one another way of proving results about Fell bundles—first prove it for groupoid crossed products, and then use Morita equivalence to transport the result to the Fell bundle setting. In this manner, I extended two of my earlier results on crossed products to Fell bundles in [LaL19].

> **Theorem 3** (LaLonde)**.** *Let G be a locally compact Hausdorff groupoid and $p : \mathcal{B} \to G$ a separable Fell bundle over G.*
>
> 1. *If the unit C^*-algebra of $\mathcal{B}$ is nuclear and G is measurewise amenable, then the Fell bundle C^*-algebra $C^*(G; \mathcal{B})$ is nuclear.*
> 2. *If the unit C^*-algebra of $\mathcal{B}$ is exact and G is exact, then $C^*(G; \mathcal{B})$ is exact.*

I also used the Stabilization Theorem of [IKSW18] to show that any exact groupoid is automatically "Fell exact", meaning that given a Fell bundle $p : \mathcal{B} \to G$ and an invariant ideal I, the associated sequence

$$0 \to C_r^*(G; \mathcal{B}_I) \to C_r^*(G; \mathcal{B}) \to C_r^*(G; \mathcal{B}^I) \to 0$$

is exact. In other words, the reduced Fell bundle functor associated to G is exact precisely when the corresponding reduced crossed product functor is exact.

The results of [LaL19] have since been used to help show that certain Fell bundle C^*-algebras satisfy the UCT [KLS22] and to study ideals in crossed products [BFPR22].

2.2 | Exact Groupoids

In the process of proving Theorem 2, it became natural to consider the notion of an *exact groupoid*. In fact, the version of the theorem that appears in [LaL15] is stated more generally in terms of the *reduced crossed product*.

Just as there are multiple definitions for the tensor product of two C^*-algebras, there is more than one way of constructing a crossed product from a C^*-dynamical system. In addition to the full crossed product (which we have been using thus far), there is a *reduced* crossed product that can be defined in a more concrete way. Unfortunately, this crossed product often behaves poorly in a functorial sense. For example, the full crossed product always preserves short exact sequences of C^*-algebras, while the reduced crossed product does not generally do so. Therefore, we declare a groupoid G to be *exact* if whenever G acts on a C^*-algebra A and $I \subseteq A$ is a G-invariant ideal, the sequence

$$0 \to I \rtimes_r G \to A \rtimes_r G \to A/I \rtimes_r G \to 0$$

of reduced crossed products is exact. This definition directly generalizes the original one for groups given by Kirchberg and Wassermann. It turns out that exactness plays an important role in several current areas of research; in particular, non-exact groupoids have provided counterexamples to the Baum-Connes conjecture for groupoids.

After defining exactness, Kirchberg and Wassermann subsequently established some permanence properties of exact groups in [KW99]. I have tried to do the same for groupoids. My first such result, which deals with *equivalence* of groupoids, appeared in [LaL16].

Theorem 4 (LaLonde). *Suppose that G and H are equivalent locally compact Hausdorff groupoids. Then G is exact if and only if H is exact.*

In short, two groupoids are *equivalent* if they admit suitably nice commuting left and right actions on a locally compact Hausdorff space. Groupoid equivalence is intimately connected with the concept of *Morita equivalence*, which is an important tool in the study of C^*-algebras. In fact, my argument mimics Katsura's [Kat04, Proposition A.10] proof that Morita equivalence preserves exactness for C^*-algebras by exploiting the *linking groupoid* (as studied by Sims and Williams in [SW12]) associated to a groupoid equivalence.

In the follow-up paper [LaL20], I established some other permanence properties of exact groupoids, mainly pertaining to subgroupoids and transformation groupoids, which model actions of groups or groupoids on spaces.

Theorem 5 (LaLonde). *Let G be an exact groupoid.*

1. *If $H \subseteq G$ is a closed, wide subgroupoid (i.e., $H^{(0)} = G^{(0)}$), then H is exact.*
2. *If $H \subseteq G$ is a closed subgroupoid and $H^{(0)}$ is an invariant subset of $G^{(0)}$, then H is exact.*
3. *If G acts on a locally compact Hausdorff space X, then the associated transformation groupoid $G \ltimes X$ is exact.*

The last result admits a partial converse, which generalizes a theorem from [KW99] about groups with cocompact exact subgroups.

Theorem 6 (LaLonde). *Let G be a locally compact Hausdorff groupoid, and suppose $H \subseteq G$ is a closed, wide, exact subgroupoid such that the map $p : G/H \to G^{(0)}$ is proper. Then G is exact.*

I also discussed some related results in [LaL20] relating to the weaker notion of *inner exactness* as defined by Anantharaman-Delaroche. In particular, inner exactness is preserved under groupoid equivalence, and an analogue of Theorem 6 holds for inner exact groupoids. I was also able to establish some results on exactness and inner exactness for groupoid extensions using Fell bundle techniques in [LaL19].

3 | Inverse Semigroups

In two recent joint papers with David Milan [LM17, LMS19], I have investigated applications of some powerful results for étale groupoids to the C^*-algebras of inverse semigroups. An *inverse semigroup* is a set S equipped with an associative binary operation such that for each $s \in S$, there is a unique element $s^* \in S$ satisfying

$$ss^*s = s, \quad s^*ss^* = s^*.$$

One can associate an étale groupoid to an inverse semigroup, which allows one to bring groupoid results to bear on inverse semigroups. As one example, the authors of [BNR+16] established a "uniqueness theorem" for the reduced C^*-algebra of a general étale groupoid G, which guarantees that a homomorphism out of $C^*_r(G)$ is injective if and only if it is injective on a certain distinguished subalgebra. In [LM17], Milan and I determined how to apply this uniqueness theorem to the reduced C^*-algebra of an inverse semigroup S using a subalgebra that is intrinsically related to the structure of S.

Theorem 7 (LaLonde-Milan). *Let S be a cryptic inverse semigroup, and $Z \subseteq S$ the centralizer of the idempotents. A homomorphism $\varphi : C^*_r(S) \to A$ is injective if and only if its restriction to $C^*_r(Z)$ is injective.*

We also obtained an analogous version of this theorem for a different C^*-algebra associated to S, called the *tight C^*-algebra*. The cryptic hypothesis is replaced with the assumption that the idempotent semilattice of S is 0-disjunctive.

Theorem 8 (LaLonde-Milan). *Let S be a 0-disjunctive inverse semigroup, and $Z \subseteq S$ the centralizer of the idempotents. A homomorphism $\varphi : C^*_{\text{tight}}(S) \to A$ is injective if and only if its restriction to $C^*_{\text{tight}}(Z)$ is injective.*

This result is perhaps more satisfying than the first, since the tight C^*-algebra of an inverse semigroup aligns more closely with other common C^*-algebraic constructions. For example, if S is the inverse semigroup of a directed graph Λ, then $C^*_{\text{tight}}(S) \cong C^*(\Lambda)$.

In our subsequent paper [LMS19] with Jamie Scott, we investigated the ideal structure of an inverse semigroup S in relation to that of its tight C^*-algebra. In particular, we determined technical hypotheses that guarantee a correspondence between certain ideals in S and closed invariant subsets of the tight filter space of S, which in turn gives a correspondence with ideals in $C^*_{\text{tight}}(S)$.

Theorem 9 (LaLonde-Milan-Scott)**.** *Let S be an inverse semigroup with idempotent semilattice E. Suppose that E satisfies the trapping condition and admits finite covers. Then there is a one-to-one correspondence between saturated invariant ideals in E and closed invariant subsets of $\widehat{E}_{\text{tight}}$.*

We also developed inverse semigroup analogues of Conditions (L) and (K) for directed graphs, which allowed us to generalize results from the study of graph C^*-algebras. Finally, we presented a stronger, but more easily verifiable, condition that implies condition (K), and we investigated some applications to the inverse semigroups of self-similar graph actions.

4 | Dynamical Systems and Pattern Avoidance

In addition to my usual work on operator algebras, Kassie Archer and I have written a paper [AL17] on discrete dynamical systems, with an eye toward applications to enumerative combinatorics. Our work focuses on the patterns that can appear when one iterates certain maps on the unit interval.

If $f : [0,1] \to [0,1]$ and $\pi \in S_n$ is a permutation, we say π is an *allowed pattern* for f if there is an $x \in [0,1]$ such that the numbers

$$x, f(x), f^2(x), \ldots, f^{n-1}(x)$$

are in the same relative order as the entries of π. Equivalently, we say π is the pattern *realized* by f at x. Not all patterns are realized by such a dynamical system; the absence of certain patterns can help one see that a sequence of allegedly random numbers was actually generated in a deterministic fashion.

In our paper, we investigated allowed patterns for the family of symmetric tent maps on $[0,1]$. We showed that if T denotes the standard tent map and T_μ is another tent map of height $\frac{1}{2} < \mu \leq 1$, then any pattern realized by T_μ is also realized by T. The proof uses analytic methods to study the behavior of "commuter functions" between T and T_μ, i.e. functions $h : [0,1] \to [0,1]$ satisfying

$$T \circ h = h \circ T_\mu. \tag{1}$$

Such functions were introduced by Bollt and Skufca in [SB08] as an attempt to relax the notion of topological conjugacy between dynamical systems. We showed that a function h satisfies (1) if and only if it is a solution to a certain functional equation, and then used the Banach Fixed Point Theorem to guarantee the existence of such a function. We then argued that such a function is strictly increasing, hence order-preserving, which allows one to translate patterns realized by T_μ into ones realized by T. We also investigated patterns that are allowed for T but forbidden for T_μ, and we established a lower bound (in terms of μ) for the length of any such pattern.

We are currently looking at related problems for certain dynamical systems. In particular, we have obtained some results on the likelihood of certain allowed patterns for *shift maps* on $[0, 1]$. We also plan to consider signed shifts (of which the tent map is a special case) and logistic maps.

5 | Undergraduate and Graduate Student Research

I am truly excited about the possibility of incorporating students in my research program. While the primary topics that I study require a broad range of background knowledge (including expertise in functional analysis and certain areas of algebra and topology) before one can conduct original research, there are some topics that would lend themselves nicely to advanced undergraduate or masters-level projects. Indeed, I have already supervised or otherwise been involved in several student projects at UT Tyler. Below are some brief descriptions of these projects, as well as some ideas for possible future work by students.

- In the summer and fall of 2017, I supervised an undergraduate student's work on the path groupoids of directed graphs. This topic is closely related to some current work on the C^*-algebras of directed graphs, but no knowledge of C^*-algebras was necessary. The student had taken our graduate-level analysis sequence, but a solid foundation in undergraduate analysis and algebra (as well as some familiarity with basic point-set topology) would have been sufficient. An advanced undergraduate could feasibly tackle similar problems that involve only the algebraic and topological properties of groupoids.

- In our paper [LMS19], David Milan and I incorporated some work that one of his REU students undertook in the summer of 2016 on algebraic properties of inverse semigroups. There are many other related questions that can be posed in this realm, and a potential student researcher would only need a strong background in undergraduate abstract algebra in order to work on them.

- The work that Kassie Archer and I have done on allowed patterns of dynamical systems is very amenable to undergraduate research. In fact, we have already had two undergraduate students work on problems related to the ones we studied in [AL17]. While a familiarity with basic enumerative combinatorics or dynamical systems might be helpful, there is no real prerequisite beyond sufficient mathematical maturity. The topic is friendly enough that a student could begin contributing right away, and learn the necessary techniques as needed.

- I have also supervised honors and senior capstone projects on topics outside my area of expertise. These project have involved modeling the firing of neurons with differential equations, elliptic curve cryptography, the isoperimetric problem, and the Axiom of

Choice. I have considerable prior experience in both applied math (namely mathematical modeling, optimization, and dynamical systems) and number theory, and I would therefore be open to supervising student projects in these areas in the future.

6 | Future Work

In relation to the projects I have just described, I have a few ideas for some possible related problems to address in the near future.

1. Moves on graph-like objects. My former undergraduate student Vincent Villalobos and I investigated moves on directed graphs (as described by Sørensen in [Sør13]) and their relationship with the associated path groupoids. Vincent gave direct proofs that all of Sørensen's moves (with the exception of the Cuntz splice) implement an equivalence between the path groupoids of the two graphs, though this fact also follows from a more general result in [CRS17] about the collapsing moves of Crisp and Gow [CG06]. Though they have now moved on to a Ph.D. program, Vincent and I would like to see if their proofs can be generalized to develop similar moves (or even general collapsing moves) for other combinatorial objects modeled by groupoids, namely inverse semigroups, higher-rank graphs, and topological graphs.

2. Exact groupoids. While I have answered most of the questions I originally set out to address regarding exact groupoids, there is still at least one problem I would like to tackle. It is well-known that a discrete group Γ is exact if and only if $C_r^*(\Gamma)$ is an exact C^*-algebra. However, it is unknown whether the analogous statement for groupoids is true: if G is an étale groupoid and $C_r^*(G)$ is exact, must G be exact? This question has been answered in the affirmative for certain classes of groupoids, and there are some partial results that connect exactness to *amenability at infinity*. However, the question remains open for arbitrary étale groupoids. This problem appears to be quite difficult in general, but it is one that I continue to think about.

3. Commuters and allowed patterns for other dynamical systems. As mentioned above, Kassie Archer and I continue to work on problems involving the likelihood of allowed patterns for certain dynamical systems. We also have plans to extend our previous work on tent maps to other discrete dynamical systems, including some preliminary work on general symmetric tent maps (i.e., we do not assume that one of them is the "standard" tent map). We would also like to revisit some of the conjectures we posed in our first paper.

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