Patternicity, Prediction, and Proof

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A Topical Joke

Question: Why did the computer science major always confuse Halloween and Christmas?

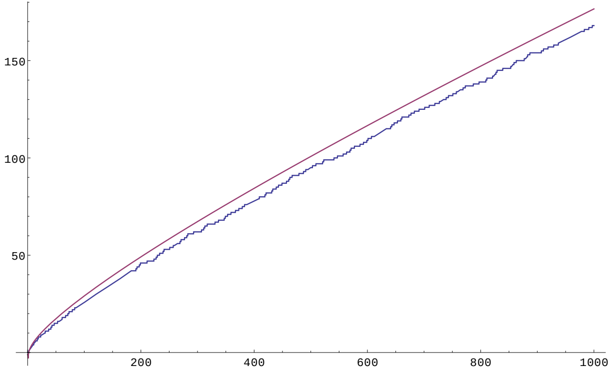
Answer: Because

$$\text{Oct } 31 = \text{Dec } 25.$$

Observation

In calculus, we often make observations about key examples to help us develop general facts and techniques. Important results often seem "obvious" (because of the functions we encounter).

Problem: Given two functions, show that one is always larger than the other. Look at a graph:



Patterniсity

Humans have a tendency to see patterns or connections between things, even when nothing meaningful is going on.

- The psychological term is **apophenia**.
- Science writer Michael Shermer coined the term **patternicity** to mean "the tendency to find meaningful patterns in meaningless noise".

We want to believe that nature is orderly, not random.

A Warm-Up

Consider the list of numbers

$$1, 2, 3, 4, 5, 6, \ldots$$

What's the next number? No, obviously it's 1000. The formula

$$a_n = n + \frac{993(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)}{6!}$$

gives $a_n = n + 0$ when $1 \leq n \leq 6$ and $a_7 = 1000$.

Okay, so there is a pattern. It's just not the one you want to find.

An Integral Pattern

It's possible to show that

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

and

$$\int_0^\infty \frac{\sin x}{x} \cdot \frac{\sin(x/3)}{x/3}\,dx = \frac{\pi}{2}.$$

Likewise,

$$\int_0^\infty \frac{\sin x}{x} \cdot \frac{\sin(x/3)}{x/3} \cdot \frac{\sin(x/5)}{x/5}\,dx = \frac{\pi}{2},$$

$$\int_0^\infty \frac{\sin x}{x} \cdot \frac{\sin(x/3)}{x/3} \cdot \frac{\sin(x/5)}{x/5} \cdot \frac{\sin(x/7)}{x/7}\,dx = \frac{\pi}{2},$$

An Integral Pattern

$$\int_0^{\infty} \frac{\sin x}{x} \cdot \frac{\sin(x/3)}{x/3} \cdots \frac{\sin(x/9)}{x/9}\, dx = \frac{\pi}{2},$$

$$\int_0^{\infty} \frac{\sin x}{x} \cdot \frac{\sin(x/3)}{x/3} \cdots \frac{\sin(x/11)}{x/11}\, dx = \frac{\pi}{2},$$

$$\int_0^{\infty} \frac{\sin x}{x} \cdot \frac{\sin(x/3)}{x/3} \cdots \frac{\sin(x/13)}{x/13}\, dx = \frac{\pi}{2},$$

$$\int_0^{\infty} \frac{\sin x}{x} \cdot \frac{\sin(x/3)}{x/3} \cdots \frac{\sin(x/15)}{x/15}\, dx = \frac{467807924713440738696537864469}{935615849440640907310521750000}\pi$$

$$= \frac{\pi}{2} - 2.31 \times 10^{-11}$$

These are called the **Borwein integrals**.

An Even Worse Integral Pattern

Look at the modified Borwein integrals

$$\int_0^\infty \frac{\sin x}{x} \cdot \frac{\sin\left(\frac{x}{101}\right)}{\frac{x}{101}} \cdot \frac{\sin\left(\frac{x}{201}\right)}{\frac{x}{201}} \cdots \frac{\sin\left(\frac{x}{100n+1}\right)}{\frac{x}{100n+1}}\, dx.$$

- These integrals all equal $\frac{\pi}{2}$ when $n < 9.8 \times 10^{42}$.
- They definitely don't equal $\frac{\pi}{2}$ when $n > 7.4 \times 10^{43}$.
- The rule is that the integral equals $\frac{\pi}{2}$ whenever

$$\sum_{k=1}^{n} \frac{1}{100k+1} \leq 1.$$

Reference: *Patterns that Eventually Fail* at `johncarlosbaez.wordpress.com`.

A Magical Formula?

Consider the function

$$f(n) = n^2 + n + 41.$$

Then

$$f(1) = 43 \qquad f(6) = 83$$
$$f(2) = 47 \qquad f(7) = 97$$
$$f(3) = 53 \qquad f(8) = 113$$
$$f(4) = 61 \qquad f(9) = 131$$
$$f(5) = 71 \qquad f(10) = 151$$

Notice anything? These numbers are all *prime*.

In fact, $f(n)$ is prime for $n \leq 39$, but not $n = 40$:

$$40^2 + 40 + 41 = 40(40 + 1) + 41 = 41^2.$$

Capricious coincidences cause careless conjectures. – Richard K. Guy

It's All Relative

Two integers n and m are **relatively prime** if $\gcd(n, m) = 1$, i.e., if n and m have no common factors other than 1.

Conjecture

The numbers $n^{17} + 9$ and $(n+1)^{17} + 9$ are relatively prime for all n.

- For $n = 1$ we get

$$1^{17} + 9 = 10 \text{ and } 2^{17} + 9 = 131081.$$

  These are relatively prime.

- Write a computer program—it'll never find a counterexample! The smallest one is

$$8424432925592889329288197322308900672459420460792433$$

Proof by lack of counterexample is not a proof at all! – Esther Arkin

Hitting for the Cycle

For each natural number n, there is a unique irreducible polynomial that is a factor of $x^n - 1$ but not a factor of $x^k - 1$ for any $k < n$. This polynomial is called the n^{th} **cyclotomic polynomial**.

$$\begin{aligned}
\Phi_1(x) &= x - 1 \\
\Phi_2(x) &= x + 1 \\
\Phi_3(x) &= x^2 + x + 1 \\
\Phi_4(x) &= x^2 + 1 \\
\Phi_5(x) &= x^4 + x^3 + x^2 + x + 1 \\
\Phi_6(x) &= x^2 - x + 1 \\
\Phi_7(x) &= x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \\
\Phi_8(x) &= x^4 + 1 \\
\Phi_9(x) &= x^6 + x^3 + 1 \\
\Phi_{10}(x) &= x^4 - x^3 + x^2 - x + 1 \\
\Phi_{11}(x) &= x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1
\end{aligned}$$

Hitting for the Cycle

Note that the coefficients are all 1 or -1 (or 0). This holds up to $n = 104$, but

$$\begin{aligned}\Phi_{105}(x) = x^{48} &+ x^{47} + x^{46} - x^{43} - x^{42} - 2x^{41} - x^{40} - x^{39} + x^{36} \\ &+ x^{35} + x^{34} + x^{33} + x^{32} + x^{31} - x^{28} - x^{26} - x^{24} - x^{22} \\ &- x^{20} + x^{17} + x^{16} + x^{15} + x^{14} + x^{13} + x^{12} - x^{9} - x^{8} \\ &- 2x^{7} - x^{6} - x^{5} + x^{2} + x + 1\end{aligned}$$

Let $A(n)$ denote the largest coefficient (in absolute value) of Φ_n.

- $A(n)$ is unbounded.
- In fact, Erdős showed that $A(n)$ is not even bounded above by any polynomial in n.

Superficial similarities spawn spurious statements. – Richard K. Guy

Gijswijt's Sequence

This sequence is defined recursively:

- Start with $a_1 = 1$.
- For each $n > 1$, form the "word"

$$a_1 a_2 a_3 \cdots a_{n-1},$$

and set a_n to be the largest number of repeating blocks at the end of this word. It's easier to see with a picture:

$$\begin{array}{lcl} 1 & \longrightarrow & a_2 = 1 \\ 11 & \longrightarrow & a_3 = 2 \\ 112 & \longrightarrow & a_4 = 1 \\ 1121 & \longrightarrow & a_5 = 1 \\ 11211 & \longrightarrow & a_6 = 2 \\ 112112 & \longrightarrow & a_7 = 2 \\ 1121122 & \longrightarrow & a_8 = 2 \end{array}$$

Gijswijt's Sequence

Keep going:

1121122	$\longrightarrow$	$a_8 = 2$
11211222	$\longrightarrow$	$a_9 = 3$
112112223	$\longrightarrow$	$a_{10} = 1$
1121122231	$\longrightarrow$	$a_{11} = 1$
11211222311	$\longrightarrow$	$a_{12} = 2$
112112223112	$\longrightarrow$	$a_{13} = 1$
1121122231121	$\longrightarrow$	$a_{14} = 1$
11211222311211	$\longrightarrow$	$a_{15} = 2$
112112223112112	$\longrightarrow$	$a_{16} = 2$
1121122231121122	$\longrightarrow$	$a_{17} = 2$
11211222311211222	$\longrightarrow$	$a_{18} = 3$

Question: When does 4 show up? Does it show up?

Gijswijt's Sequence

The number 4 does appear—the first time is at $n = \mathbf{220}$.

The creators of this sequence initially conjectured that no number larger than 4 would ever appear.

- 5 does appear, somewhere around $n = 10^{10^{23}}$.
- The sequence is **unbounded**.

References: *A Slow-Growing Sequence Defined by an Unusual Recurrence*, `http://arxiv.org/abs/math/0602498`

OEIS – Sequence A090822

An Oddity

Pólya's Conjecture, 1919

For each natural number n, there are more natural numbers less than n with an odd number of prime factors than there are with an even number of prime factors.

- Define $\Omega(k)$ to be the number of prime factors of k, counted with multiplicity. For example,

$$\Omega(3^2 \cdot 5 \cdot 11^4) = 7.$$

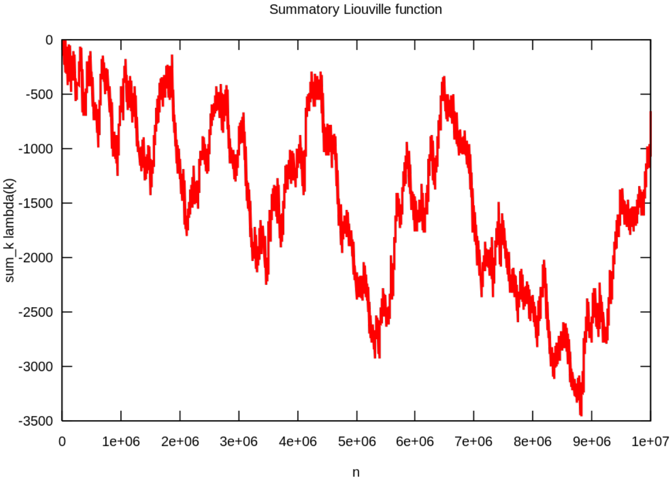
- Now set

$$L(n) = \sum_{k=1}^{n} (-1)^{\Omega(k)}.$$

  Then Pólya's conjecture is equivalent to saying that $L(n) \leq 0$ for all n.

An Oddity

Here's a graph of $L(n)$ for $0 \leq n \leq 10^7$:



An Oddity

In 1958, Haselgrove showed there must be a counterexample to Pólya's conjecture. He estimated it to be around

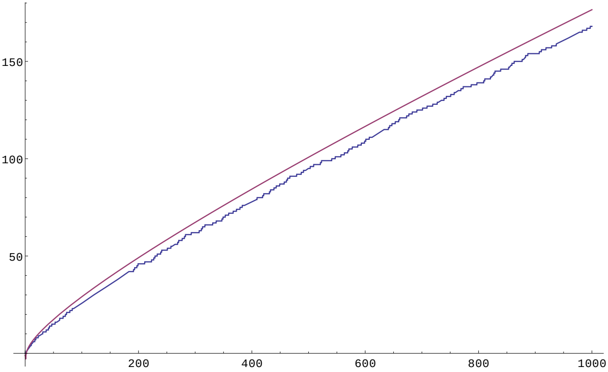
$$n = 1.845 \times 10^{361}.$$

The smallest counterexample is now known to be 906,150,257.

You can't tell by looking. – Richard K. Guy

Hope Clouds Observation

Problem: Show one function is always larger than the other.



Top: $\text{Li}(x) = \int_2^x \frac{1}{\ln t}\, dt$

Bottom: $\pi(x) = \#\{p \le x : p \text{ is prime}\}$

Li(x) versus $\pi(x)$

Theorem (Prime Number Theorem)

$$\lim_{x \to \infty} \frac{\pi(x)}{Li(x)} = 1$$

In other words, $\pi(x)$ and Li(x) "asymptotically" behave the same.

Conjecture

Li(x) $\geq \pi(x)$ for all x.

Littlewood showed that this conjecture is false. (In fact, he showed the two functions cross infinitely often!)

Skewes showed the first crossing occurs for some $x < 10^{10^{10^{964}}}$.

This has since been improved to 1.397×10^{316}.

Patternicity vs. Proof

Why is it that the Prime Number Theorem is true, but the conjecture that $\mathrm{Li}(x) \geq \pi(x)$ is false, when we had numerical evidence for both?

We can *prove* the Prime Number Theorem!

Just because we make an observation or check a conjecture for a large number of cases, we can't say it's true without giving a **proof**.

Two Current Problems

The Riemann Hypothesis

All of the "nontrivial" zeros of the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s \in \mathbb{C} \setminus \{1\}$$

lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$.

RH has been checked for billions of zeros up to $\mathrm{Re}(s) = 10^{20}$.

It carries a $1 million Millennium Prize.

Some Current Problems

The Collatz Conjecture

Define the function

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ 3n+1 & \text{if } n \text{ is odd} \end{cases}$$

Now choose a number a_1 and apply the function repeatedly:

$$a_2 = f(a_1), a_3 = f(a_2), \ldots$$

Eventually you will reach 1, regardless of the starting point.

The Collatz Conjecture has been checked for all $a_1 < 87 \times 2^{60}$.

It carries a \$500 prize, due to Erdős, though he said "Mathematics may not be ready for such problems."

This is an extraordinarily difficult problem, completely out of reach of present-day mathematics. – Jeff Lagarias

There aren't enough small numbers to meet the many demands made of them.

See *The Strong Law of Large Numbers* by Richard K. Guy