# Stokes and the Surveyor's Shoelaces

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UT Tyler Math Club

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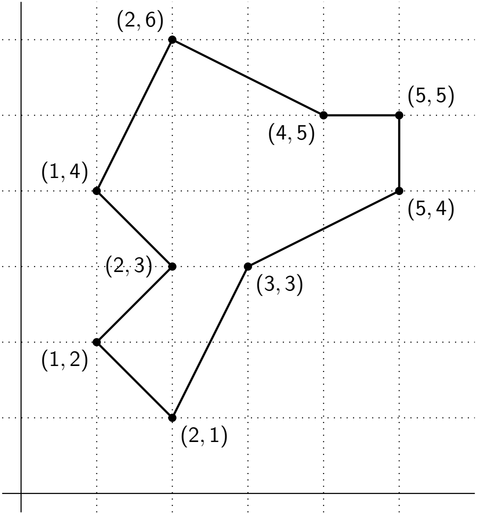
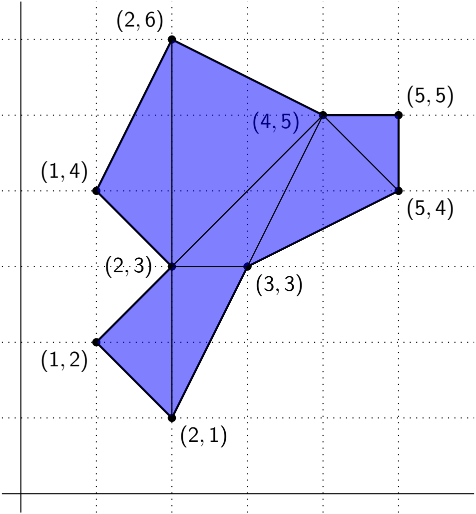
# Finding Areas of Polygons

**Problem:** Is there a way to quickly find the area of a polygon just by knowing where its vertices are?



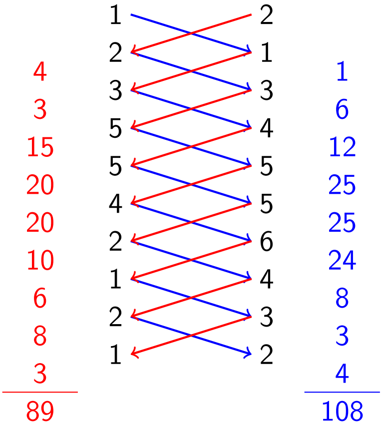
# Finding Areas of Polygons

**One approach:** "Triangulate" the polygon.




$$A = 1 + 1 + \frac{3}{2} + 3 + 1 + \frac{3}{2} + \frac{1}{2} = \boxed{\frac{19}{2}}$$

I want an easier way. List out the coordinates of the points in counterclockwise order. Then "cross multiply".



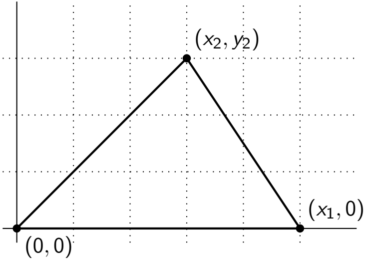

Now add up both columns, subtract the left from the right, and divide by 2:

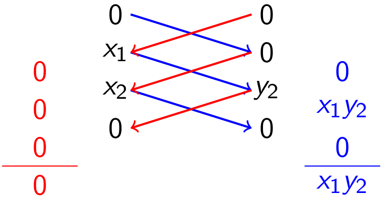
$$\frac{1}{2}(108 - 89) = \boxed{\frac{19}{2}}$$

That's the area!

# Areas of Triangles

How do we know this algorithm works in general? Let's check it first for **triangles**. First assume things look like this:

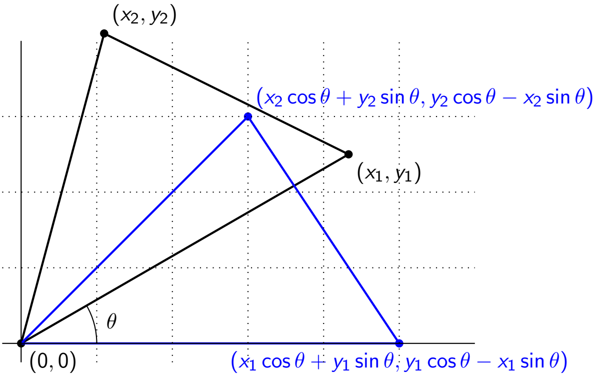



It works, since the area is $\boxed{A = \frac{1}{2} x_1 y_2}$.

# Areas of Triangles

Now suppose our triangle isn't situated so nicely. Then *rotate* it.




$$\begin{aligned}
2A &= (x_1 \cos\theta + y_1 \sin\theta)(y_2 \cos\theta - x_2 \sin\theta) - (x_2 \cos\theta + y_2 \sin\theta)(y_1 \cos\theta - x_1 \sin\theta) \\
&= x_1 y_2 \cos^2\theta - x_1 x_2 \sin\theta\cos\theta + y_1 y_2 \sin\theta\cos\theta - x_2 y_1 \sin^2\theta \\
&\qquad - x_2 y_1 \cos^2\theta + x_1 x_2 \sin\theta\cos\theta - y_1 y_2 \sin\theta\cos\theta + x_1 y_2 \sin^2\theta \\
&= x_1 y_2 - x_2 y_1
\end{aligned}$$

# Areas of Triangles

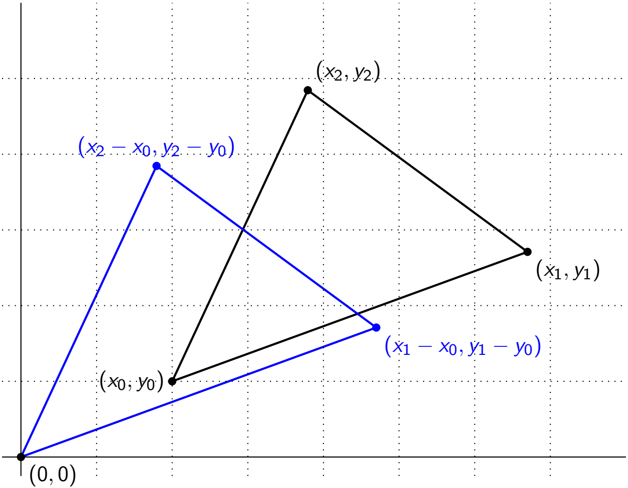
There's a shorthand notation for this—it's an example of a **determinant**.

$$A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = \tfrac{1}{2}(x_1y_2 - x_2y_1).$$

You may have seen this sort of area formula in one of your classes that uses matrices.

# Areas of Triangles

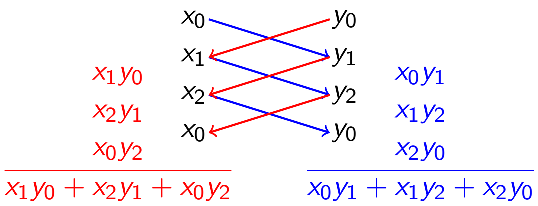
Finally, what if the triangle isn't based at the origin? Then *shift* it.



Now use the determinant formula:

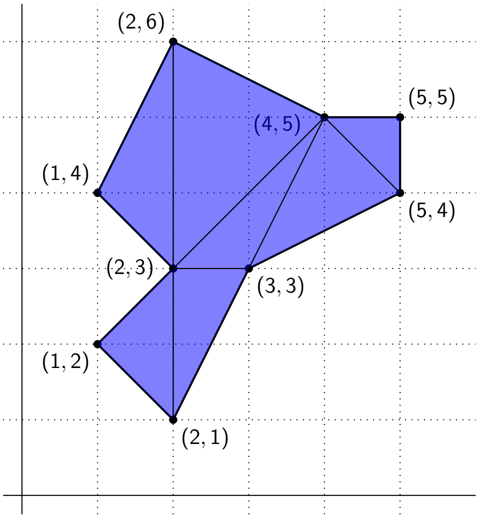
$$A = \frac{1}{2}\begin{vmatrix} x_1 - x_0 & y_1 - y_0 \\ x_2 - x_0 & y_2 - y_0 \end{vmatrix}$$
$$= \tfrac{1}{2}\left[x_1y_2 - x_1y_0 - x_0y_2 + x_0y_0 - x_2y_1 + x_2y_0 + x_0y_1 - x_0y_0\right]$$
$$= \tfrac{1}{2}\left[x_0y_1 + x_1y_2 + x_2y_0 - x_1y_0 - x_0y_2 - x_2y_1\right].$$

This is exactly what we get from the Shoelace Formula:



# Proof by Example

Our first example illustrates how the general proof would work.




$$
\begin{aligned}
&1\cdot 1 + 2\cdot 3 + 2\cdot 2\\
&-2\cdot 2 - 1\cdot 2 - 3\cdot 1\\
&+2\cdot 3 + 3\cdot 3 + 2\cdot 1\\
&-1\cdot 3 - 3\cdot 2 - 3\cdot 2\\
&+2\cdot 3 + 3\cdot 5 + 4\cdot 3\\
&-3\cdot 3 - 3\cdot 4 - 5\cdot 2\\
&+2\cdot 5 + 4\cdot 6 + 2\cdot 3\\
&-3\cdot 4 - 5\cdot 2 - 6\cdot 2\\
&+2\cdot 6 + 2\cdot 4 + 1\cdot 3\\
&-3\cdot 2 - 6\cdot 1 - 4\cdot 2\\
&+3\cdot 4 + 5\cdot 5 + 4\cdot 3\\
&-3\cdot 5 - 4\cdot 4 - 5\cdot 3\\
&+5\cdot 5 + 5\cdot 5 + 4\cdot 3\\
&-4\cdot 5 - 5\cdot 4 - 5\cdot 3
\end{aligned}
$$

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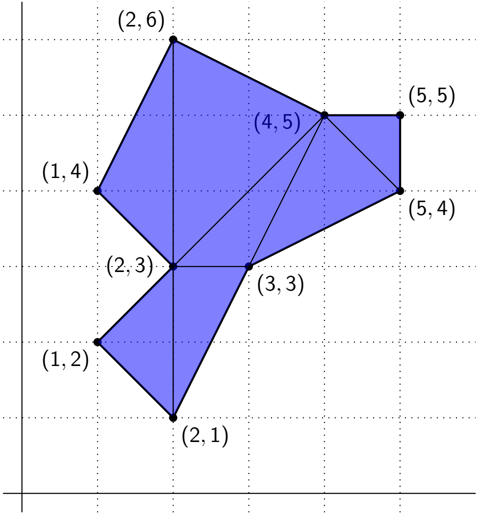
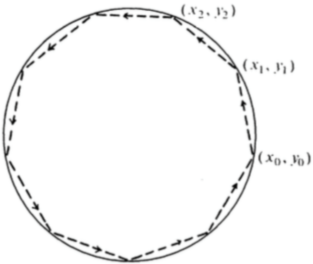



$$
\begin{aligned}
&1 \cdot 1 + 2 \cdot 3 + 2 \cdot 2 \\
&- 2 \cdot 2 - 1 \cdot 2 - 3 \cdot 1 \\
&+ 2 \cdot 3 + 3 \cdot 3 + 2 \cdot 1 \\
&- 1 \cdot 3 - 3 \cdot 2 - 3 \cdot 2 \\
&+ 2 \cdot 3 + 3 \cdot 5 + 4 \cdot 3 \\
&- 3 \cdot 3 - 3 \cdot 4 - 5 \cdot 2 \\
&+ 2 \cdot 5 + 4 \cdot 6 + 2 \cdot 3 \\
&- 3 \cdot 4 - 5 \cdot 2 - 6 \cdot 2 \\
&+ 2 \cdot 6 + 2 \cdot 4 + 1 \cdot 3 \\
&- 3 \cdot 2 - 6 \cdot 1 - 4 \cdot 2 \\
&+ 3 \cdot 4 + 5 \cdot 5 + 4 \cdot 3 \\
&- 3 \cdot 5 - 4 \cdot 4 - 5 \cdot 3 \\
&+ 5 \cdot 5 + 5 \cdot 5 + 4 \cdot 4 \\
&- 4 \cdot 5 - 5 \cdot 4 - 5 \cdot 5
\end{aligned}
$$

# Approximating Areas

Can we use the Shoelace Formula to estimate the areas of more complicated shapes? Yes—approximate the shape with a polygon!




Let's put this idea to use: `https://acme.com/planimeter/`

We should expect that using a polygon with more sides yields a better approximation to the area.

Letting the number of sides $\to \infty$ should give some sort of integral.

# Approximating Areas

Rewrite the shoelace formula as follows:

$$A = \frac{1}{2}\left[\begin{vmatrix} x_0 & y_0 \\ x_1 & y_1 \end{vmatrix} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \cdots + \begin{vmatrix} x_{n-1} & y_{n-1} \\ x_n & y_n \end{vmatrix} + \begin{vmatrix} x_n & y_n \\ x_0 & y_0 \end{vmatrix}\right]$$

For each $i$, let $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_i - y_{i-1}$. It's easy to check that

$$\begin{vmatrix} x_{i-1} & y_{i-1} \\ x_i & y_i \end{vmatrix} = \begin{vmatrix} x_{i-1} & y_{i-1} \\ \Delta x_i & \Delta y_i \end{vmatrix}$$

Now use the Shoelace Formula to build "Riemann sums":

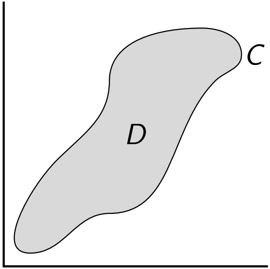
$$\lim_{n\to\infty} \frac{1}{2}\sum_{i=1}^{n} \begin{vmatrix} x_{i-1} & y_{i-1} \\ \Delta x_i & \Delta y_i \end{vmatrix} = \frac{1}{2}\oint_C \begin{vmatrix} x & y \\ dx & dy \end{vmatrix} = \boxed{\frac{1}{2}\oint_C x\,dy - y\,dx}$$

This is a special case of **Green's theorem**.

# Green's Theorem

**Idea:** An integral over a region $D$ can be done by computing a different integral along the boundary $C$.

$$\iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \oint_C P\,dx + Q\,dy$$



# Stokes's Theorem

Green's theorem (and hence the Shoelace Formula) is a special case of a more general theorem about integrals and boundaries.

**Theorem (Stokes)**

*Suppose $M$ is an $n$-dimensional smooth, oriented manifold with boundary and $\omega$ is an $(n-1)$-form on $M$, then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

Implications:

- The Fundamental Theorem of Calculus follows from Stokes's theorem.
- The Divergence Theorem and Stokes's Theorem from vector calculus are special cases.

# The End

You can read more about the Shoelace Formula in:

- Bart Braden, *The Surveyor's Area Formula*, The College Mathematics Journal **17**(4), 1986.


From *Saturday Morning Breakfast Cereal.*