Zero Coupon Price Theory

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Treasury STRIPS

Consider the pricing of zero coupon Treasury STRIPS derived from the interest coupon portion of the bond. These bonds are formed by taking the coupon interest of a Treasury bond due on a specific payment date and creating a zero coupon bond which matures on that date. The STRIPS are structured so that all bonds derived from the interest portion maturing at the same date are given the same CUSIP identifier and are indistinguishable from each other even if they originally came from different bonds. The Treasury allows the primary dealers to create the STRIPS and they can also reconstitute Treasury bonds from the STRIPS, but to remake a bond, the original principal STRIP, must be combined with the appropriate maturity values for each of its coupons from the coupon STRIPS market. For this reason, the trading of the STRIPS derived from the principal portion of the bond, might be valued differently than a coupon STRIP of the same maturity date; the principal STRIPS each have a unique CUSIP identifier.

For what follows we consider only the coupon STRIPS market, as each security has only one defining characteristic and that is its waiting time until maturity. There may be even in this market some supply and demand problems for the long term STRIPS, because all the waiting times over 20 years must come from the Treasury issuing 30 year bonds. Should the supply of 30 year bonds be inadequate for demand then pricing anomalies could occur. We will assume the supply of the longer bonds is adequate and that there should not be pricing anomalies based on separate supply demand issues regarding longer waiting times. This is essential for the model, which assumes an asymptotic relationship to a long term rate over time.

Price Term Structure

Thus in the model t = 0, is maturity time, where P(0) = 1. All P(t) for t > 0 are less than 1. What follows is a model for the price structure of the zero coupon Treasury bond based on an ordinary differential equation. In the discussion, rates are continuously compounded annual rates and the time scale is in years.

P(t) is the bond price function and differential equation is shown below with the condition that P(0) is 1.

$$P(0) = 1$$

$$P'(t) = -r(t)\, P(t)$$

where t ≥ 0, r(t) ≥0

Further for the bond price we would also like

$$\lim_{t\to\infty} r(t) = r, \;\; a \text{ constant},$$

which is the maximum rate and the asymptotic slope of the logarithmic plot of bond prices versus time to maturity.

The function also needs a minimum value,

$r(0) = \text{r0}$

With this set up one possible equation structure for r(t) might be,

$r(t) = \text{r0} + (r - \text{r0})\, F(t)$

where F(t) could be any probability distribution function on the domain: t ≥ 0.

Thus we have an ordinary differential equation where r(t) is not related to P(t). This generally will need the actual function r(t) to find a solution.

Before pursuing this structure further, consider the standard discount formula for pricing a zero coupon bond with a maturity value of 1. If there is a continuously compounded interest rate parameter, r, then

$P(t) = e^{-\text{r0}\, t}$

This is the solution for the differential equation if r(t) is a constant, r0. For the situation where r0 is not constant, the following formula is chosen for r(t):

$r(t) = \mathrm{r0} + (r - \mathrm{r0})\,\mathrm{CDF}[\mathrm{GammaDistribution}[k, 1], t]$, or

$$r(t) = \mathrm{r0} + (r - \mathrm{r0})\left(\begin{cases} 1 - \frac{\Gamma(k,\, t)}{\Gamma(k)} & t > 0 \\ 0 & \text{True} \end{cases}\right)$$

where Γ is the gamma function.

Mathematica can find a solution for this equation as follows. r0 is changed to r as a constant; the function r(t) will not be used further.

```
DSolve[{P[0] == 1, P'[t] == -(r0 + (r - r0) (1 - Gamma[k, t]/Gamma[k])) P[t]}, P[t], t]
FullSimplify[%[[1]][[1]][[2]]]
```

$$\left\{\left\{P[t] \to e^{-r\,t + \frac{r\,t\,\mathrm{Gamma}[k,t]}{\mathrm{Gamma}[k]} - \frac{\mathrm{r0}\,t\,\mathrm{Gamma}[k,t]}{\mathrm{Gamma}[k]} + \frac{r\,\mathrm{Gamma}[1+k,0]}{\mathrm{Gamma}[k]} - \frac{\mathrm{r0}\,\mathrm{Gamma}[1+k,0]}{\mathrm{Gamma}[k]} - \frac{r\,\mathrm{Gamma}[1+k,t]}{\mathrm{Gamma}[k]} + \frac{\mathrm{r0}\,\mathrm{Gamma}[1+k,t]}{\mathrm{Gamma}[k]}}\right\}\right\}$$

$$e^{-r\,t + \frac{(r-\mathrm{r0})\,(t\,\mathrm{Gamma}[k,t] + \mathrm{Gamma}[1+k,0] - \mathrm{Gamma}[1+k,t])}{\mathrm{Gamma}[k]}}$$

```
Limit[e^(-r t + ((r - r0) (t Gamma[k, t] + Gamma[1 + k, 0] - Gamma[1 + k, t]))/Gamma[k]), k -> 0]
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$$e^{-r\,t}$$

There is a limit as k → 0, so the full formula becomes.

$$P(t) = \begin{cases} e^{-r\,t} & t \ge 0 \wedge k = 0 \\ e^{-\frac{(r-\mathrm{r0})\,(t\,\Gamma(k,t) - \Gamma(k+1,t) + \Gamma(k+1,0))}{\Gamma(k)} + r\,t} & t > 0 \wedge \mathrm{r0} \ge 0 \wedge r \ge \mathrm{r0} \\ 1 & \text{True} \end{cases}$$

Since the yield for a zero coupon bond can be calculated by:

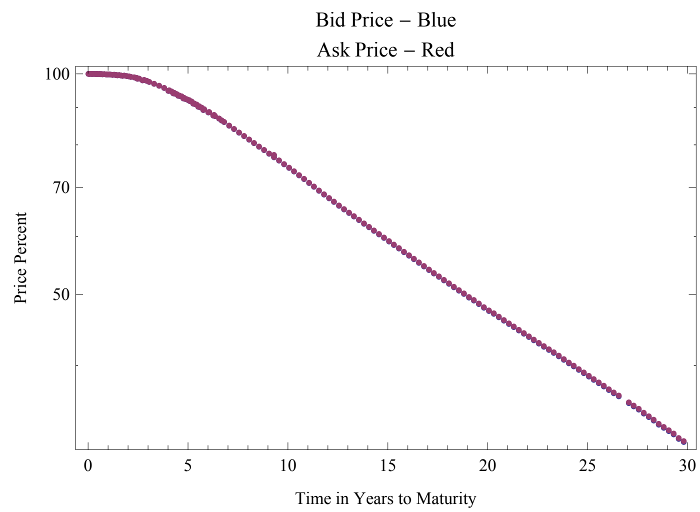
$$\mathrm{yield}(t) = -\frac{\log(P(t))}{t}$$

The final yield formula becomes:

$$\mathrm{yield}(t) = \begin{cases} r & t \ge 0 \wedge k = 0 \\ \frac{r\,t - \frac{(r-\mathrm{r0})\,(-\Gamma(k+1,t) + t\,\Gamma(k,t) + \Gamma(k+1,0))}{\Gamma(k)}}{t} & t > 0 \wedge \mathrm{r0} \ge 0 \wedge r \ge \mathrm{r0} \\ \mathrm{r0} & \text{True} \end{cases}$$

■ Fitting Data

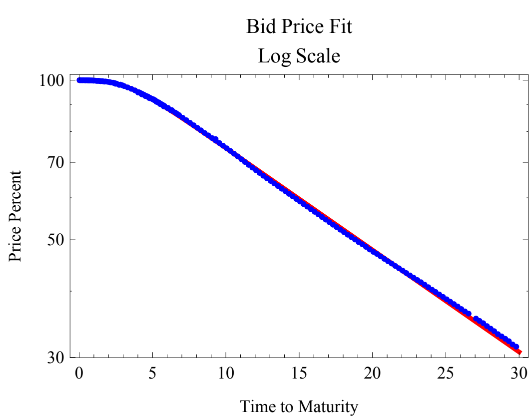
Next look at some actual data. The graph below shows prices in percent for a recent series of maturities of coupon STRIPS as published daily in *The Wall Street Journal*, on 22 January 2014. The scale is logarithmic and the long end of the waiting times to maturity is nearly linear.

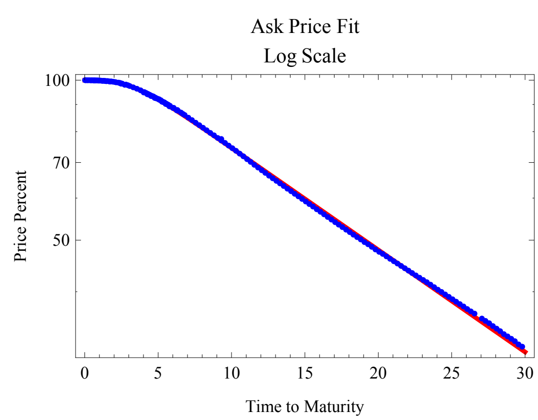


The linear portion of the curve is consistent with the exponential price model describe above, when k is 0. Since the price is in percent, a price of 100 is equivalent to a value of 1 at maturity. The bend in the price data curve as t moves closer to zero is, however, not consistent with the exponential model, which should be linear on a logarithmic plot all the way to zero. The k parameter in the equation determines the shape of the bend.

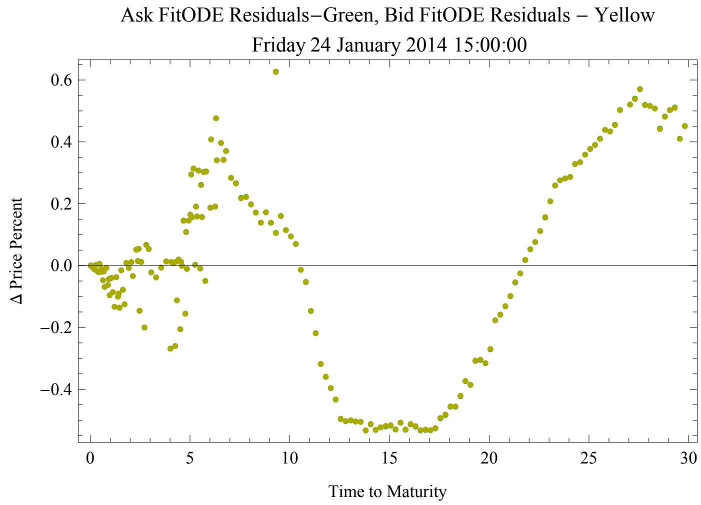
The fits to this model are good, with respect to the location and major bend in the price curve. It should be noted that there are some things this model cannot fit. It has only one inflection point and cannot support oscillations in the distal yield or price curve, that are supported by empirical methods to fit the data. One needs to question, however, whether the waves in the yield curve are completely rational. The model approaches a long term continuous rate asymptotically without oscillation, so if there is an oscillation in the data, the curve will be placed in the center of it by the fitting algorithm. The function being used was selected to be most accurate for current data.

The red curve below is the fit and the blue dots are the individual prices for zero coupon bonds shown above. The oscillation in the long term prices on this kind of plot seems subtle, but it becomes more apparent on the yield plot. Zero coupon bonds in the 25 to 30 year to maturity range are priced higher (have lower yield) than those in the 12 to 20 year range. Part of this effect may be due to supply and demand with relative shortage of long term maturities, but this doesn’t explain the dip in the 12 to 20 year range which may be due to the use of an interpolating formula to determine the yield curve from broader Treasury bond trading, such that the zero coupon bonds are priced by dealers using a formula.





The pattern of oscillation is best seen in the fit residuals for both bid and ask fits.



The statistics for the bid and ask parameter fits are shown below.

	Estimate	Standard Error	t-Statistic	P-Value
r0	0.000701838	0.0004783	1.46736	0.144126
r	0.0443088	0.0000755429	586.538	5.56802×10^{-283}
k	3.41406	0.049586	68.8514	4.32554×10^{-126}

	Estimate	Standard Error	t-Statistic	P-Value
r0	0.000600327	0.00047897	1.25337	0.211792
r	0.0442101	0.0000755269	585.356	7.84557×10^{-283}
k	3.41375	0.0496369	68.7745	5.19558×10^{-126}

■ Interpretation of Parameters

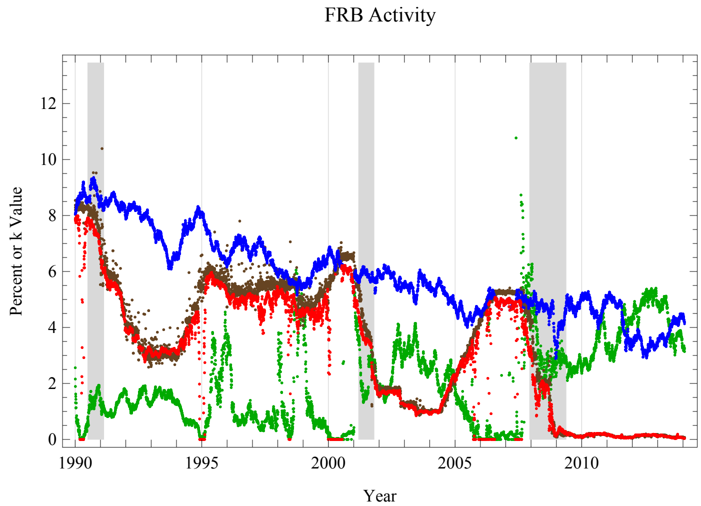
The model can thus fit the general shape of current data with only three parameters and it can be used to study how the shape has changed over time. The r0 parameter is the minimum continuous rate at t = 0. The r parameter represents the maximal slope of the linear portion of the log price plot or the asymptotic long term rate on the yield curve. k is a measure of the degree of bending of the log price plot, which would be linear at k = 0, and it also determines the more complicated shape of the yield curve. The model was designed primarily to be used for zero coupon bond prices. The yield interpretation from the model is therefore based on a continuous yield model and it may not be ideal for yield curves based on coupon bearing bonds as will be shown below. There is a key assumption and that is that as time to maturity increases, the bond price should always be falling and consequently yield should be rising. The model thus cannot fit an inverted or humped yield curve. The best interpretation it can give for this situation is k = 0 and constant r, slicing through an average region of the curve..

■ Influence of Federal Reserve Policy

To study this model over time, daily yield curves from the Treasury website dating back to 1990 were analyzed. The rates were converted to continuous rates by the formula:

$$\text{continuousRate} = \log\left(1 + \frac{\text{percentRate}}{100}\right)$$

then parameters were determined by fitting data to the yield(t) formula. For charting the rate parameters, r0 and r, were converted back to effective annual rates in percent, to match the daily Fed funds time series. The graph below shows the interesting results.



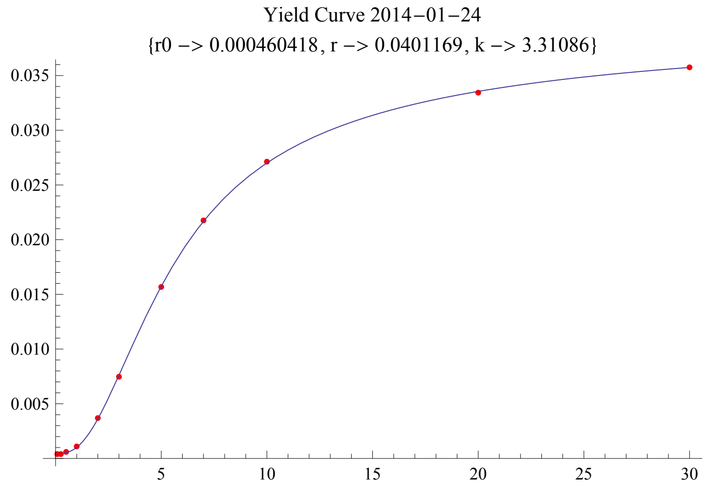
In the graph, r, the asymptotic rate from the formula, is shown in blue. The Fed Funds time series, DFF, obtained from the FRED database is shown in brown. The r0 rate is in red and closely follows the Fed funds rate in brown. The k parameter, which determines the shape of the yield curve, is in green. The light gray background shows periods of officially declared recessions.

While looking at the plots, one needs to keep in mind that the theoretical model does not support an inverted yield curve, nor a curve containing a large hump. The best the function can do is set, k to zero during the fitting. r0 is constrained also and cannot be greater than r. So when the yield curve is inverted, ideally the fit should show r0 = r and k = 0, however, if the data points are very slightly and slowly increasing, the curve fitting can create a high value for k and a value for r0 significantly lower than r. These represent bad fits to data which are not fully consistent with the model, but there has been no attempt to remove them from the plot. They can generally be identified as those points which don't match the main trend of each plot and occur when k should equal 0.

As the Fed funds rate increases, the k parameter decreases and the yield curve flattens. While the Fed is easing, k rises bending the short end of the yield curve more than the long end. When k is zero the model yield curve is perfectly flat, but the actual data points could be declining or humped with a declining trend. That does not imply that the Fed is no longer affecting the market. Each time on the chart the Fed funds rate becomes higher than the r parameter and the main path of k has been near zero for a while, a recession has followed. Notice particularly the sustained short period of the elevated Fed funds rate in 2000 and in 2006 - 2007, in both cases recessions quickly followed. Prior to this chart there was also a brief period where the Fed funds were higher than the r curve in 1989, just before the 1990-1991 recession. The r0 rate in red follows the Fed funds rate in brown. It is hard to conclude from the chart that the FOMC didn't play some role in causing recessions. Longer histories of yield curves shown in the appendix show the same pattern for all the recessions in the last fifty years.

■ Discussion

The model was designed primarily to work with zero coupon Treasury bond prices (as shown in the Fitting Data section), which have only one characteristic--their time to maturity. So it is a little surprising to see very good fits to an interpolated yield curve representing primarily coupon bearing bonds. The fits should continue to be good into the future unless the yield curve becomes inverted again. If the Fed uses a gradual withdrawal strategy, decline in the k parameter should be useful to monitor the effect of the Fed's manipulation of rates.

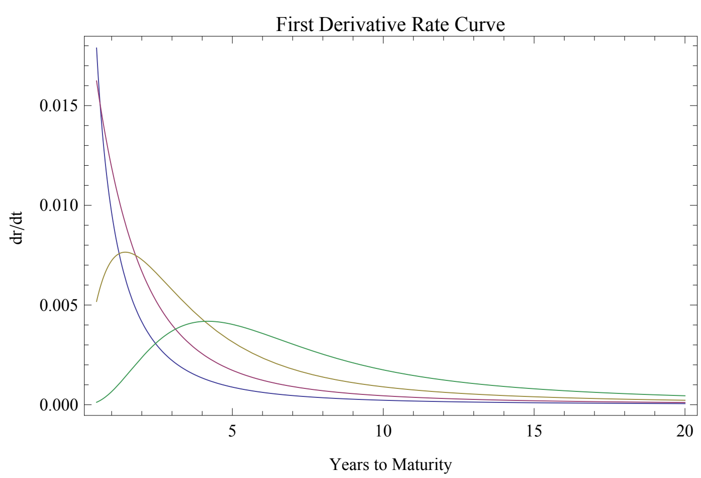


	Estimate	Standard Error	t- Statistic	P- Value
r0	0.000460418	0.0000506159	9.09633	0.000017135
r	0.0401169	0.0000903825	443.857	7.43383×10^{-19}
k	3.31086	0.0148059	223.618	1.79026×10^{-16}

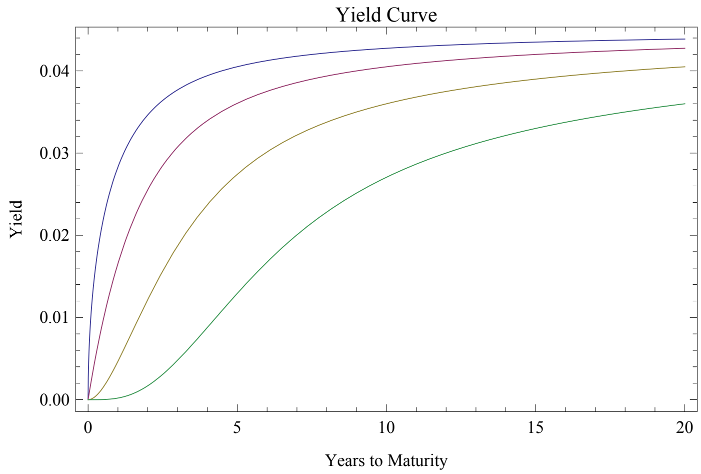
When k is 0, the yield in the model is constant across all times, this is the flattest curve that the model can measure. The model supports only a yield curve where the yield is either flat or always rising toward r. If the model is given an inverted yield curve, it should find k close to 0 and r will approximate the mean rate, while $r0 \to r$.

At $k \leq 1$, the behavior of the function is different from $k > 1$. For $k \leq 1$, the rate function is always increasing, but it has the highest rate of increase at $t = 0$, and the rate of increase of the rate function decreases as t increases. For $k > 1$, there is an inflection point and the first derivative of the yield curve is still positive definite, but it rises to a peak then falls, with the peak coming later in t as k increases.

The first derivative formula below shows this perhaps a little more clearly. The blue, red, yellow, and green curves represent values of k, 0.5, 1, 2, 4, respectively



Below are the corresponding yield curves.



History shows that the bend in the yield curve in U.S. markets did not occur before central bank intervention during the Great Depression[1]. The term structure of rates thus is not part of human nature, but rather it is induced by an interaction between markets and the central bank. The Federal Reserve Bank has never before deformed the yield curve so dramatically. Hopefully this model will provide a useful tool to monitor the withdrawal of quantitative easing and future rate tightening. During periods of tightening, the k parameter is a measure the flattening of the yield curve. If the Fed funds rate is pushed above the asymptotic rate, r, the previous experience in the FRB Activity chart above and those in the appendix show a recession has always followed when k also has been near zero. Using this tool the FOMC may be able to prevent recessions, by not letting the k parameter fall below 1.0 when it is tightening. It is an experiment worth trying.

References

1. Homer, S, Sylla, R, *A History of Interest Rates*, Fourth Edition, Wiley, 2005.

A. 2. Gürkaynak R., Sack B, Wright, JH, The U.S. Treasury Yield Curve: 1961 to the Present. Federal Reserve Discussion Series: 2006-28, http://www.federalreserve.gov/Pubs/feds/2006/200628/index.html

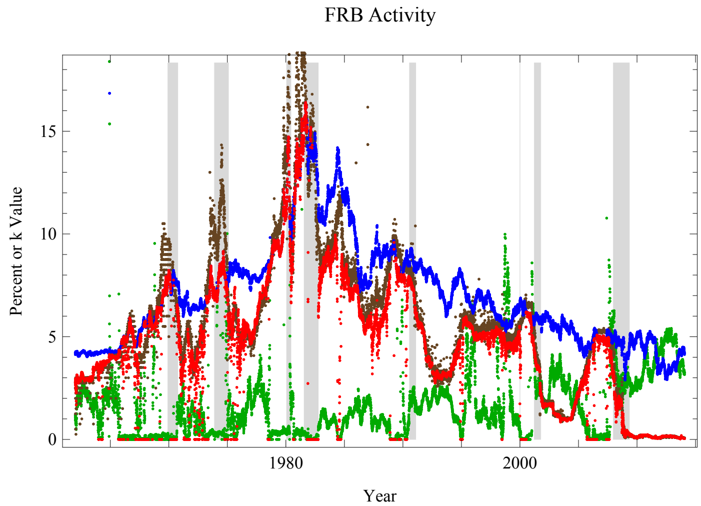
B. 3. Federal Reserve Bank of St. Louis, FRED DFF time-series: http://research.stlouisfed.org/fred2/graph/?id=DFF

C. 4. Jordan, BD, , Randy D. Jorgensen, RD, Kuipers, DR, The relative pricing of U.S. Treasury STRIPS: empirical evidence. Journal of Financial Economics 56 (2000) 89}123

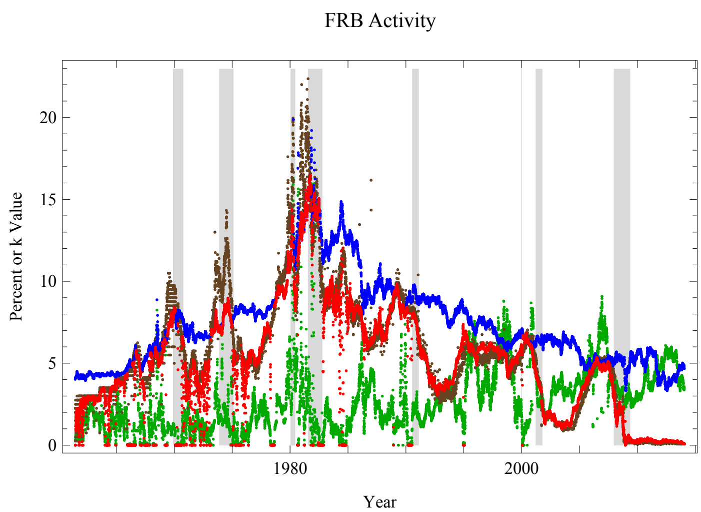
D. 5. *Mathematica,* http://www.wolfram.com.

■ Appendix

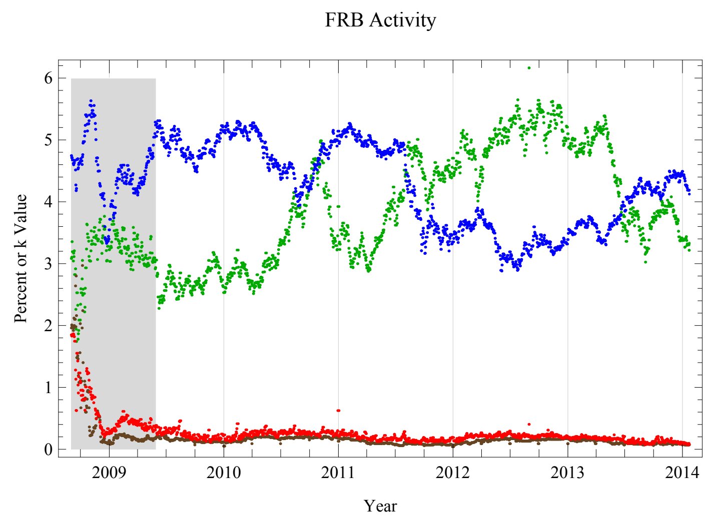
The yield formula has been tested against a longer yield curve series, which has been reconstructed from the daily constant maturity Treasury rate series on the FRED database. The layout of the graph is the same as above recessions in gray are preceded by Fed funds rates in brown which exceed the r rate.



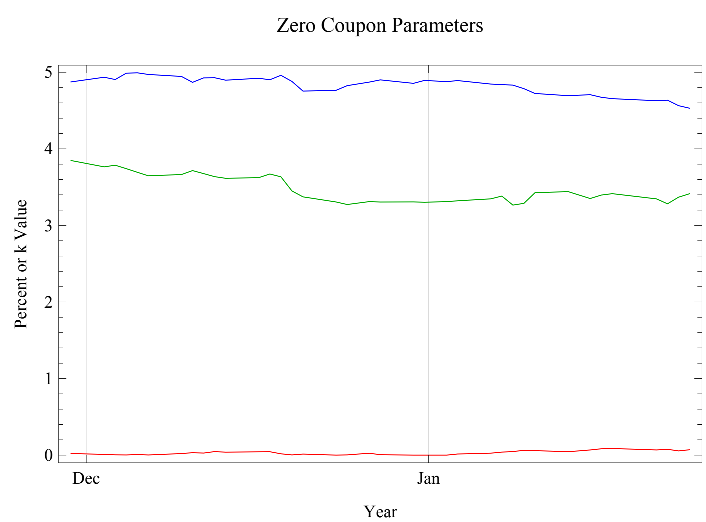
The data below, SVENY01 to SVENY30, were from the data set provided by the Federal Reserve. These data represent a fit to Treasury bond data, adjusted to zero coupon rates using a formula and methods described in a working paper at the link above. This series has in general a few more interpolated points, but the curves have some odd shapes which may in part be due to the interpolation operation. They were harder to fit to the model than the other series.



This data set is from TreasuryDirect database, consisting of bonds which have been sold in the past directly from the Treasury. TreasuryDirect doesn't sell STRIPS, so the only zero coupon bonds in this mix are Treasury bills with maturity of one year or less. There are earlier prices in the series, but they don't have the maturity date for the bonds, only the CUSIP which would have to be looked up to find the date. To analyze the data continuous yields were calculated from the end of day bond price and time to maturity, then fit by the formula.



The data published daily in the WSJ for Treasury STRIPS is being collected prospectively since the beginning of data. The fit to the bid prices is shown below. r0 - red, r - blue, k - green. Only the coupon interest strips are fit, but these data have more points than yield curves so the fits based on prices should be more accurate as the equation was derived to model zero coupon bond prices rather than yield curves based on coupon bearing bonds. These securities have been available since the mid 1980s and historical price series should be available from CRSP, which has quoted a base price of $1,000 plus labor without a subscription. It should also be available to those with a subscription to WRDS. The company which supplies the data to the WSJ did not respond to an email, but they have supplied data to other researchers. I am happy to help anyone with access to data to do the fitting. The STRIPS carry a higher r, rate than the coupon bearing bonds above, this is presumably to offset the higher rate risk of a zero coupon bond. The k parameter looks comparable but a longer series will be needed.



Links

Link to *Mathematica* source page.

Link to interactive module.

Wikipedia list of recessions.

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