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NOTE

Chord Distributions for Shape Matching

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A shape matching method based on concepts tied to integral geometry is studied. The method consists of computing the distribution of random chords over the figures to be matched and comparing them by means of the Kolmogorov–Smirnov test. A simplified form of the procedure for matching figures to a circle yields a test for circularity. It is generalized to obtain a statistical test for shape matching and then modified to produce a dissimilarity measure between shapes. The circularity test is performed on 30 shapes and compared to two other circularity measures found in the literature. Experiments are also performed using the matching procedures.

1. INTRODUCTION

Shape is a concept that has an intuitive appeal; however, attempts to quantify shape for the analysis of images have had only limited success [1]. In this paper, we assume a "shape" is a simply connected compact region in a two-dimensional Euclidean space, which may or may not be convex. The set of boundary or frontier points of the region will be used to characterize its shape.

Shape matching attempts to assess the proximity of two or more shapes. A shape matching procedure can yield a proximity measure between shapes or it can implement the predicate SIMILAR. The similarity predicate or the proximity measure should have invariance, or at least insensitivity, to changes in the size, the position, and the rotation of the figures to which it is applied. Also, for digitized shapes, the effect of the digitization resolution should be minimized. A further useful feature is the ability to simulate human performance in shape discrimination, though this is a feature that is inherently ambiguous and difficult to evaluate.

A shape matching procedure that is by its definition position, rotation, and size invariant is presented in this paper. Also, for digital figures, it appears to be insensitive to digitization resolution. The method yields both a statistical test for similarity and, with modification a dissimilarity measure between shapes. First, we present background literature. Then we present a simplified form of the shape matching method which matches shapes to the form of a circle, thus generating a test and a measure of circularity. The method is expanded to yield a test for the similarity of two digitized figures and is then modified to give a dissimilarity measure between shapes. Next, the data set of shapes on which experiments were performed is described. We then give the experiments along with their results. The first experiment compares the circularity test with two other measures described in the literature. The next involves testing the country outlines in the data set for

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similarity. The final experiment involves computing a dissimilarity measure between the country outlines. The paper is then concluded with a summary of the results.

2. RELATED WORK

One way to look at shape matching is based on concepts taken from integral geometry [2–8]. Integral geometric techniques use a set of measuring devices M from which a sample is drawn at random. For shape analysis, M is usually either the set of all lines in the plane or the set of all oriented chords (finite length line segments) in the plane. Integral geometry concerns itself with defining the concept of randomness, so that objects, when measured by their interaction with elements of M, will have the same measure (in the measure theoretic sense) under a given group of transformations [2]. For shape analysis, this group is taken to be the group of rigid-body motions. It can then be shown that the "$dp\,d\theta$ differential" is the only invariant measure for lines in the plane, where θ is the angle of the perpendicular to the line with the x axis, and p is the distance of the line from the origin [2].

Next, given a geometric property F, integral geometry addresses the problem of computing various statistics of (the frequency distribution of the values of) F, for subsets S of the manifold in which M is embedded. For shape analysis, S would be the test shape, while F could be defined to be any interesting property of the intersection of S with elements of M. As an example, let M be the set of all lines in the plane, so that $L(p, \theta) \in M$ is the line at distance p from the origin, where the perpendicular to the line makes an angle θ with the x axis. If we let $F(S \cap L(p, \theta))$ be the length of $S \cap L(p, \theta)$, then the mean of the distribution, i.e., $\iint F(S \cap L(p, \theta))\, dp\, d\theta$, can be shown to be the area of S.

Novikoff [3] defines the basic ideas needed to apply the paradigm of integral geometry to pattern recognition. Tenery [4] uses these ideas to define $P(d)$, which is the conditional probability that if the origin of a directed line segment of length d falls within an object S, then the terminal point of the segment also falls within S. Tenery claims that this distribution should characterize the "shape" of S. One problem is, of course, the computation of $P(d)$. Tenery computes $P(d)$ by sampling 1000 points at random and numerically approximating $P(d)$ for several figures. Another problem is to compare the $P(d)$ functions for two shapes in shape matching. Here, Tenery chooses to use the mean-square vector differences of the two $P(d)$ functions at incremental values of d. His results show that $P(d)$ performs quite well in shape discrimination.

Wong and Steppe [5] define the measuring device to be the set of all infinite straight lines and define F to be the length of the intersection of a line with S. This corresponds to the example presented above. They set up a number of two-class decision problems, and use Wald's sequential probability ratio test for classifying a given test shape. This allows them to take as many random lines as necessary for discrimination with a preset error probability. Their results show fairly good discrimination, although the number of samples needed to discriminate between two similar shapes is very large. Mallows and Clark [6], however, are able to show that these distributions of lengths of intersections with random lines can be equivalent for two different (nonidentical) shapes.

Moore [7, 8] presents a method for extracting features to describe shapes based on two functions, the metric pattern function and the angle pattern function, which use the concept of random chords. These are related to integrals of the autocorrelation of the member function defining Tenery's probability distribution. Moore shows that global measures relating to these two functions cluster shapes in a reasonable way.

3. PROPOSED CIRCULARITY TEST

We propose using an easily implementable technique for shape matching based on the above ideas. As in Wong and Steppe's method, our technique utilizes the distribution of lengths of random chords over a figure. We define a random chord, however, by choosing two points at random from the boundary of a shape. This way of choosing random chords further complicates the problem of computing the distribution of chord lengths over arbitrary sets [9]. For digital pictures, however, we have a finite set of boundary points from which we can randomly choose two to define the end points of our chord, and so one can easily compute the empirical distribution of chord lengths. Also, the exact distribution of these chord lengths over a circle is known [10]. A direct method for computing this distribution is shown in the Appendix.

We first simplify the matching procedure to match a given figure with the simple parametric shape of the circle. The test for circularity of a digital figure with n points along its boundary is derived by computing the empirical distribution of distances between points chosen at random along the boundary and testing for equality with the known theoretical distribution using the Kolmogorov–Smirnov (K–S) test of goodness of fit. Two problems arise with this procedure. First, care must be taken in choosing the sample of distances between boundary points to be included in computing the sample distribution. This is because the K–S test assumes that the samples from which the empirical distribution is formed are independent. Note that for a digital figure with n boundary points, all $n(n-1)/2$ interpoint distances or chord lengths are not independent. Hence, in computing the sample distribution, we are forced to choose randomly and without replacement at most $[n/2]$ pairs of boundary points and include only the corresponding $[n/2]$ distances in our sample, where $[\cdot]$ denotes the greatest integer function.

A more fundamental problem concerns the selection of a member of the family of theoretical distributions of interpoint distances for the K–S test, since the family is indexed by the parameter r, the radius of the circle. The radius must be estimated from the given boundary points of a digital figure. We used the longest length in the sample set of $[n/2]$ distances to estimate this parameter.

Since the $[n/2]$ distances used to compute the sample distribution may not adequately characterize the given figure, we generated ten different sets of $[n/2]$ distances using a Monte Carlo procedure. The K–S test is applied to each of these 10 empirical distributions. We define a measure of circularity using the 10 tests as $-\ln(\text{average } p \text{ value})$, where the p value for each of the tests is the critical value for the test, i.e., the lowest allowable error in the test that would result in the rejection of the null hypothesis (the equality of the distributions) under the K–S test. The quantity $-\ln(\text{average } p \text{ value})$ gives a measure of circularity with limiting values of zero for a perfect circle and infinity for a perfect "noncircle." Alternatively, the number of rejections of the null hypothesis among the 10 K–S tests may be used to characterize the circularity of a test figure. By increasing the number of Monte Carlo runs, a more accurate estimate of the circularity measure should be obtained.

4. PROPOSED SHAPE MATCHING METHODS

The shape similarity test for two given digitized shapes is a generalization of the circularity test. If the two shapes have respective boundary sizes of n and m points, the empirical distributions of chord lengths are formed using $[n/2]$ and $[m/2]$ randomly selected pairs. Size invariance is again a problem and the two distributions are scaled so that the perimeter of the figure from which they arose would have a length of one unit. A two-sample K–S test is performed on the two resulting distributions. Again, 10 Monte Carlo runs of the procedure are performed to ensure adequate information is retained about the shape of each figure.

To use all the information present in each of the figures an ad hoc shape matching method that produces shape dissimilarities can also be defined. The empirical distributions are computed but with the independence assumption among the random chords violated. Thus for figures with m and n boundary points, $\binom{m}{2} - m$ and $\binom{n}{2} - n$ distances are included in the chord distributions, respectively. Distances along the perimeter are excluded since these arise from digitization or smoothing. The distributions are normalized by the perimeters of their respective figures and the value of the K–S test statistic is computed between the two normalized chord distributions. The K–S test can no longer be applied since the distribution of its statistic under the null hypothesis with dependent samples is unknown, but the statistics value can be used as a shape dissimilarity measure.

5. DATA SET

We now present the test figures used in the experiments.

5.1. Circles with Equidistant Points: CIRU

Three hundred equidistant points are generated on the perimeter of a circle (radius = 100, 500, 1000).

5.2. Circles with Random Points: CIRR

Here, 300 points are generated on the perimeter of a circle (radius = 100, 500, 1000) with angles from the center chosen randomly from the interval $[0, 2\pi)$.

5.3. Noisy Circles: CV1 *and* CV2

To test for the robustness of the circularity measures in the presence of "noise," noisy circles were generated in the following manner. Points are generated as in 5.1 and Gaussian noise with mean zero and variance σ^2 is added to the distance from the center of the circle to every point on the boundary. The variance σ^2 equals V times the radius of the circle, $V = 1, 2$. See Fig. 1 for a circle of radius 100 with $V = 1$ and Fig. 2 for a circle of radius 100 and $V = 2$ generated in this manner. Again, the radii of 100, 500, and 1000 were used to test for size invariance and effects of digitization.

5.4. Equilateral Triangles: TRI

Here, 300 equidistant points are generated on the perimeter of an equilateral triangle with the radius of the circumscribing circle set to 100, 500, and 1000.

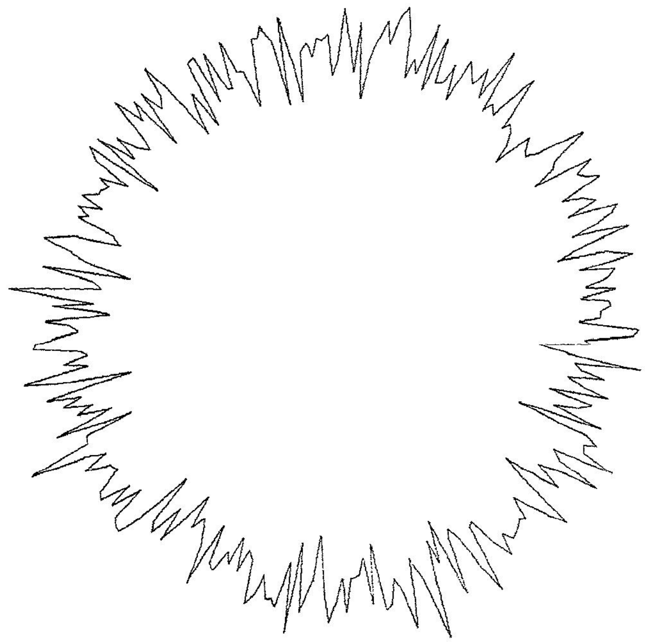

FIG. 1. Noisy circle, radius = 100, $V = 1$.

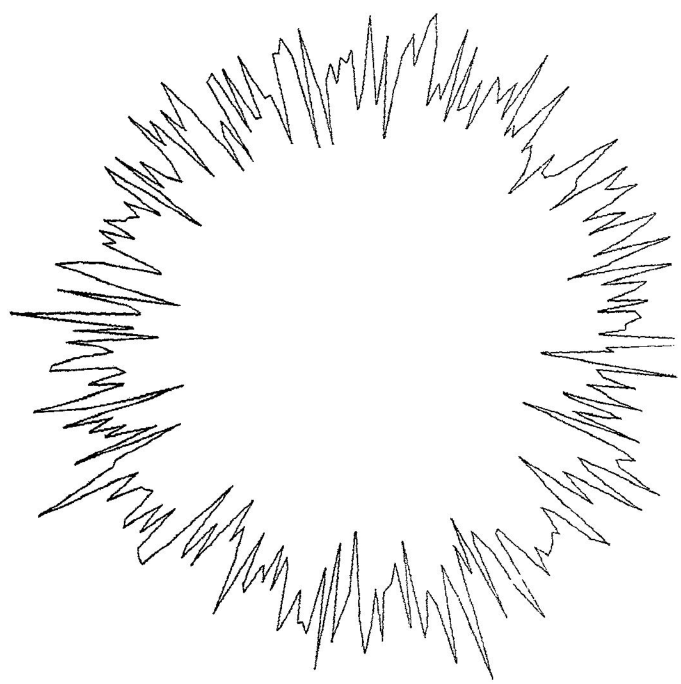

FIG. 2. Noisy circle, radius = 100, $V = 2$.

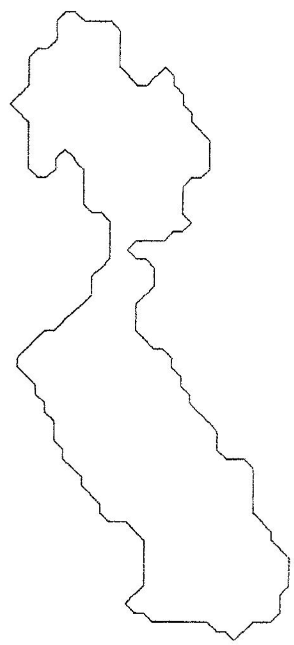

FIG. 3. Digitized boundary for SHAPE1.

5.5. *Octagon:* OCTA

This figure is a digital octagon with 39 pixels on each side, for a total of 304 points on the perimeter.

5.6. *Muscle Cells:* SHAPE1 *to* SHAPE6

These are the digitized boundaries of muscle cells described in [11]. The number of points on the perimeters of SHAPE1–SHAPE6 are 128, 193, 191, 87, 338, 58, respectively. They are shown in Figs. 3–8.

5.7. *Country Outlines*

These are the digitized boundaries of England, Italy, India, and Cuba. The border of each country has been digitized twice, once at low resolution and once at high

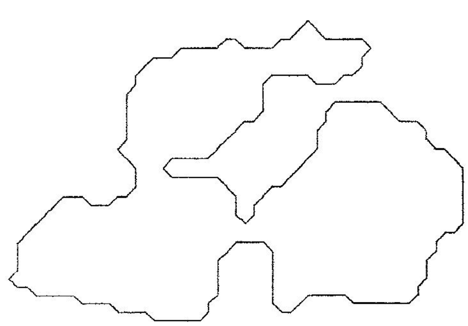

FIG. 4. Digitized boundary for SHAPE2.

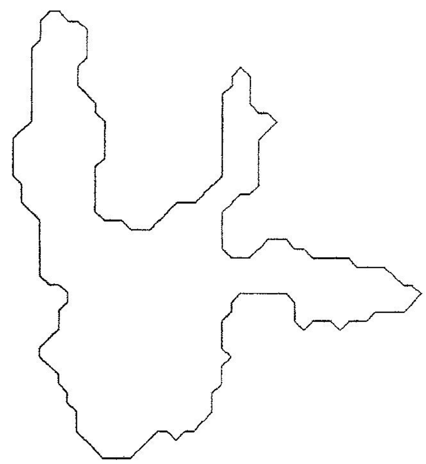

FIG. 5. Digitized boundary for SHAPE3.

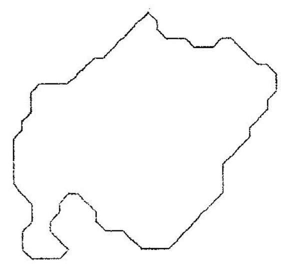

FIG. 6. Digitized boundary for SHAPE4.

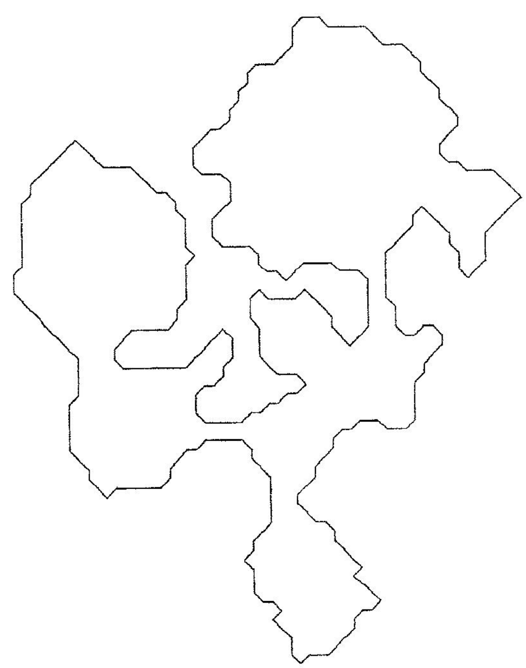

FIG. 7. Digitized boundary for SHAPE5.

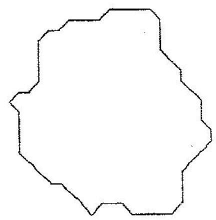

FIG. 8. Digitized boundary for SHAPE6.

resolution. England1, Italy1, India1, and Cuba1 have 482, 292, 346, and 296 points, respectively, whereas England2, Italy2, India2, and Cuba2 have 715, 343, 457, and 737 points, respectively.

6. EXPERIMENT 1

Experiment 1 consists of comparing the results from the proposed circularity test to two measures of circularity given in the literature. The first of these measures is the classic dispersion measure $P^2/4\pi A$ [12, 13]. The isoperimetric inequality states that for any closed planar figure with perimeter P and area A, $P^2/A \geqslant 4\pi$, with equality if and only if the figure is a circle. The other measure is Haralick's M/σ [14], the ratio of the mean of chords from the center of a figure to its boundary points to the standard deviation of these distances. A circle should have an infinite value of M/σ since the variance of these distances should be zero.

The results of applying each of the circularity measures to the data set are listed in Table 1. For the random chord method, we also list the number of rejections among the 10 K–S tests at a 0.95 level of confidence. For the two sets of "perfect" circles CIRU and CIRR, Haralick's measure tends to assume widely varying values as the size of the circle is changed, probably because this measure is infinity for a "perfect" circle. This could be eliminated by taking the logarithm of the values of the measure. Also, Haralick's measure determines noisy circles (CV1 and CV2) to be more circular than a perfect circle (CIRR) of radius 500. Because in CIRR the points are randomly chosen on the boundary of the perfect circle, it is more likely that the estimated center would have substantial error. This would result in large variance in the distance of the boundary points from the center, thus decreasing M/σ. In the case of CV1 and CV2, the points are equally spaced around the center and the above problem does not occur. The dispersion measure and the random chord measure label CIRR more circular than CV1 and CV2.

Even though the shape of the triangle data (TRI) substantially differs from that of a circle, the dispersion measure does not assign significantly different values to them. In fact, this measure considers noisy circles (CV1 and CV2) to be less circular than triangles.

Note that the measure of circularity of the octagon (1.060) is in the same range of values as for the two sets of circles (1.046 for CIRR and 1.038 for CIRU) based on the dispersion measure. The octagon is also considered to be more circular than CIRR based on Haralick's circularity measure. This again is due to the fact that M/σ is more sensitive to how points are chosen on the boundary. The octagon, however, is discriminated well from circles using the random chord measure.

TABLE 1

The Three Circularity Measures for the 30 Shapes

	Test data set	Dispersion	Haralick's M/σ	Random chord Rej.	Random chord $-\ln(\text{avg})$
1	CIRU 100	1.038	353.030	1	1.146
2	500	1.001	1153.275	1	1.164
3	1000	1.001	999.423	1	1.371
4	CIRR 100	1.046	22.854	0	1.112
5	500	1.003	9.691	1	1.204
6	1000	1.002	23.747	1	1.119
7	CV1 100	33.317	9.747	10	9.885
8	500	8.119	18.168	7	4.337
9	1000	4.958	21.057	7	3.689
10	CV2 100	62.620	7.413	10	12.262
11	500	14.532	16.698	10	8.154
12	1000	8.120	18.629	8	4.601
13	TRI 100	1.730	4.439	10	13.685
14	500	1.660	4.466	10	10.180
15	1000	1.655	4.470	10	14.830
16	OCTA	1.060	30.919	7	3.775
17	SHAPE1	3.903	2.078	10	18.984
18	SHAPE2	4.530	2.408	10	23.239
19	SHAPE3	4.939	2.626	10	21.685
20	SHAPE4	1.868	4.747	10	6.174
21	SHAPE5	8.329	2.136	10	43.366
22	SHAPE6	1.326	9.477	0	1.887
23	England1	8.488	2.318	10	54.004
24	England2	9.552	2.370	10	88.661
25	Italy1	4.949	2.520	10	22.121
26	Italy2	5.433	2.423	10	34.347
27	India1	4.086	3.524	10	25.286
28	India2	4.165	3.535	10	33.288
29	Cuba1	7.193	2.014	10	39.686
30	Cuba2	11.529	2.043	10	41.234

For the muscle cells (SHAPE1–SHAPE6), we can produce a ranking of the cells based on each measure's estimate of circularity. This yields the following ranking (from most circular to least circular):

Dispersion : $6 > 4 > 1 > 2 > 3 > 5$,

Haralick's M/σ : $6 > 4 > 3 > 2 > 5 > 1$,

Random chord method: $6 > 4 > 1 > 3 > 2 > 5$.

The three circularity measures tend to agree for the more circular shapes (SHAPE6 and SHAPE4), but Haralick's measure diverges from the other two after that. Referring to SHAPE1 and SHAPE3 (Figs. 3 and 5) one can see that Haralick's measure seems to judge elongated shapes as less circular than shapes which contain a circular body and sharp protrusions. Haralick's measure also ranks SHAPE1 as the least circular shape over SHAPE5, the least circular shape for the other two measures. Again, this may be due to the hypothesis stated above.

The results of the performance of each measure on the country border data may be compared. Haralick's measure gives the best results, in terms of giving the most stable values for the same shape digitized at different resolutions. Note, however, that the random chord measure depends on the number of points sampled along the boundary, since this number is used in computing the significance probability from the K–S test value. As the number of boundary points increases, more information is available to make the decision on the circularity of a given shape. Of course, for infinitely sampled shapes, this decision will be either circular (giving a measure of zero) or not circular (giving a measure of infinity). For instance, if the number of points on the boundary of CIRR with radius 1000 is increased from 300 to 600 points, the measure of circularity computed by the random cord method decreases from 1.119 to 0.938. Though this circularity measure approaches zero for a perfect circle asymptotically, the empirical results show that values close to 1 imply that the given test figure is circular.

We can also compare the computational requirements of each of the three measures. The dispersion measure requires the computation of the perimeter and the area of the figure, both of which can be computed in order(n) time and order(n) space, where n is the number of boundary points. Haralick's circularity measure depends on the computation of the center of a digital figure and on the computation of the mean and the variance of the distances from the center to each boundary point, both of which can be done in order(n) time and order(n) space. The random chord method requires the computation of $[n/2]$ interpoint distances and the computation of the K–S test statistic. The $[n/2]$ distances require order(n) time and order(n) space. Our implementation of the K–S test statistic takes order($n \log n$) time and order(n) space, and this can be reduced to order(n) time [15].

One can assess the circularity of a given digital figure by looking at the numerical values assigned to it by each of the three circularity measures studied here. If one is interested in only making a simple decision, however, such as circular versus noncircular, then a threshold must be established for each type of measure. It appears that the number of rejections of the K–S tests using the random chord method is more suitable for this type of decision. As Table 1 indicates at most one rejection of the null hypothesis of circularity is made among the 10 K–S tests for the circular figures (CIRU, CIRR, and SHAPE6) while the K–S test is rejected at least 7 times for noncircular figures. The number of rejections, of course, depends on the number of Monte Carlo runs. Other experiments, however, indicate that the ratio of the number of rejections of the K–S test to the total number of K–S tests remains less than ten percent for circular figures and more than 70% for noncircular figures as the number of Monte Carlo runs is increased. For example, out of 50 K–S tests, 49 were rejected for SHAPE4, 50 for SHAPE5, 3 for CIRU with a radius of 1000, and 1 for CIRR with a radius of 1000.

7. EXPERIMENT 2

We performed the shape similarity test using the $[n/2]$ random chord method on the country outlines. The number of acceptances out of 10 Monte Carlo trials with a test of size 0.05 are shown in Table 2. The whole dissimilarity matrix is shown so that each pair of distinct country outlines is tested twice. This gives some idea of the variance of the number of acceptances. We note that the pair of outlines for each country is accepted as similar (minimum number of acceptances is 8 out of 10)

TABLE 2

Results of the Similarity Test on the Country Outlines[a]

	England1	England2	Italy1	Italy2	India1	India2	Cuba1	Cuba2
England1	10	10	0	0	0	0	0	0
England2	9	9	0	0	0	0	0	0
Italy1	0	0	9	8	7	3	10	2
Italy2	0	0	10	10	8	5	6	5
India1	0	0	5	7	9	10	0	3
India2	0	0	4	4	10	10	0	3
Cuba1	0	0	8	8	1	0	10	2
Cuba2	0	0	6	2	2	2	3	10

[a]Entries are the number of acceptances out of 10.

except for the two outlines representing Cuba. This error is probably due to the digitization of Cuba2 which contained considerable boundary noise because of its high resolution and so adversely effected the perimeter normalization of its chord distribution. A mediocre separation, however, between the Italy outlines and those of India and Cuba can be observed.

In terms of the variance of the number of acceptances, the largest difference occurs for the pair of outlines Italy1 and Cuba2 which had acceptances of 6 out of 10 and 2 out of 10. This seems reasonable for such a statistical test.

8. EXPERIMENT 3

Experiment 3 looks at the shape dissimilarities computed by using all $\binom{n}{2} - n$ interpoint distances for a figure with n boundary points. Again the country outline data is used, but to cut down on sorting time for the K–S test all figures are smoothed so that there are only 50 boundary points. The upper triangular dissimilarity matrix that results is shown in Table 3. The entries in the matrix are the value of the K–S statistic times 10^{-1}. Here, the outlines representing the same country are well matched, with the largest dissimilarity of 0.012 between India1 and India2. The smallest dissimilarity between different countries is 0.057 between Italy2 and Cuba2. The countries seem well separated except that Italy and Cuba may be considered moderately close.

TABLE 3

Dissimilarities between Smoothed Country Outlines[a]

	England2	Italy1	Italy2	India1	India2	Cuba1	Cuba2
England1	0.22	1.45	1.41	1.56	1.64	1.78	1.75
England2		1.48	1.48	1.48	1.62	1.86	1.79
Italy1			0.12	1.27	1.24	0.63	0.63
Italy2				1.28	1.26	0.59	0.57
India1					0.25	1.62	1.63
India2						1.61	1.59
Cuba1							0.14

[a]Entries are values of the K–S statistic $\times 10^{-1}$.

9. SUMMARY

We have viewed shape matching as matching chord distributions. This yields both statistical tests for the similarity of figures and also dissimilarities between shapes. The circularity test presented appears to perform well against circularity measures found in the literature. The expanded form of the test, used to test figures for similarity, appears weak in distinguishing differing forms. To alleviate this problem, the method is modified to yield shape dissimilarities which appear reasonable on the modest experimental figures.

Future work in the area should include a theoretical analysis of the shape discrimination ability of these types of chord distributions. Can radically different shapes generate the same chord distribution? Also, the ad hoc method using all the interpoint distances should be tried with some other measure of distance of the distributions rather than the sup-norm used in the K–S statistic.

APPENDIX

Analytical Distribution of the Distance between Two Points Chosen Uniformly on the Boundary of a Circle

By symmetry, we look at the distribution of distances in half a circle, and, without loss of generality, we assume that one point is fixed at $(0,0)$ and the other point (x, y) is placed randomly on the boundary of the circle (see Fig. 9). It is easy to show by using a simple transformation that placing a point (x, y) uniformly on the boundary of the circle is equivalent to assuming that θ is uniform on the interval $[0, \pi/2]$, where θ is the angle which the chord joining points $(0,0)$ and (x, y) makes with the X axis. Now, for a circle of radius r, the length of the chord is

$$d = 2r\cos(\pi/2 - \theta) = 2r\sin(\theta), \qquad 0 \leqslant \theta \leqslant \pi/2.$$

So we get

$$\frac{\partial d}{\partial \theta} = 2r\cos(\theta).$$

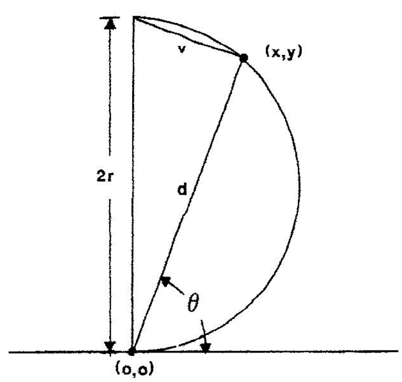

FIG. 9. Point (x, y) is placed randomly on the half circle to define the distance d.

Therefore, for the appropriate intervals of d, θ, and u,

$$f_D(d) = f_\theta(u)/(2r\cos(\theta)),$$

where $f_D(d)$ is the density of a random chord of length d, and $f_\theta(u)$ is the density of θ which is uniform on the interval $[0, \pi/2]$. It is easy to verify from Fig. 9 that

$$2r\sin(\pi/2 - \theta) = v = \left((2r)^2 - (d)^2\right)^{1/2}$$

or

$$\cos(\theta) = \sin(\pi/2 - \theta) = \left((2r)^2 - (d)^2\right)^{1/2}/2r.$$

Therefore,

$$\begin{aligned} f_D(d) &= (2/\pi)\left((2r)^2 - (d)^2\right)^{-1/2} && 0 \leqslant d \leqslant 2r, \\ &= 0, && \text{otherwise,} \end{aligned}$$

and the corresponding distribution function is

$$\begin{aligned} F_D(d) &= 0, && d < 0, \\ &= (2/\pi)\sin^{-1}(d/2r), && 0 \leqslant d \leqslant 2r, \\ &= 1, && d > 2r. \end{aligned}$$

See Coleman [10] for details of how this definition of a random chord relates to other possible definitions.

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